Chaos and the Reparametrization Mode on the AdS_2 String

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Seminar at ITMP, Lomonosov Moscow State University

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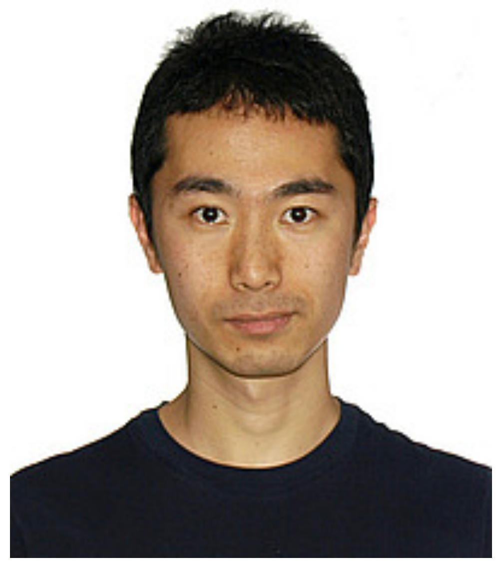

based on arXiv:2212.14842
(with Simone Giombi (Princeton) and Shota Komatsu (CERN))

Outline

Part I: Wilson loops/strings correspondence and the Wilson loop defect CFT

Part II: OTOCs and scattering amplitudes on the AdS_2 string

Part III: Computing OTOCs in conformal gauge via the reparametrization mode

Part I

Wilson loops/strings correspondence

and the Wilson loop defect CFT

Wilson loops in gauge theory

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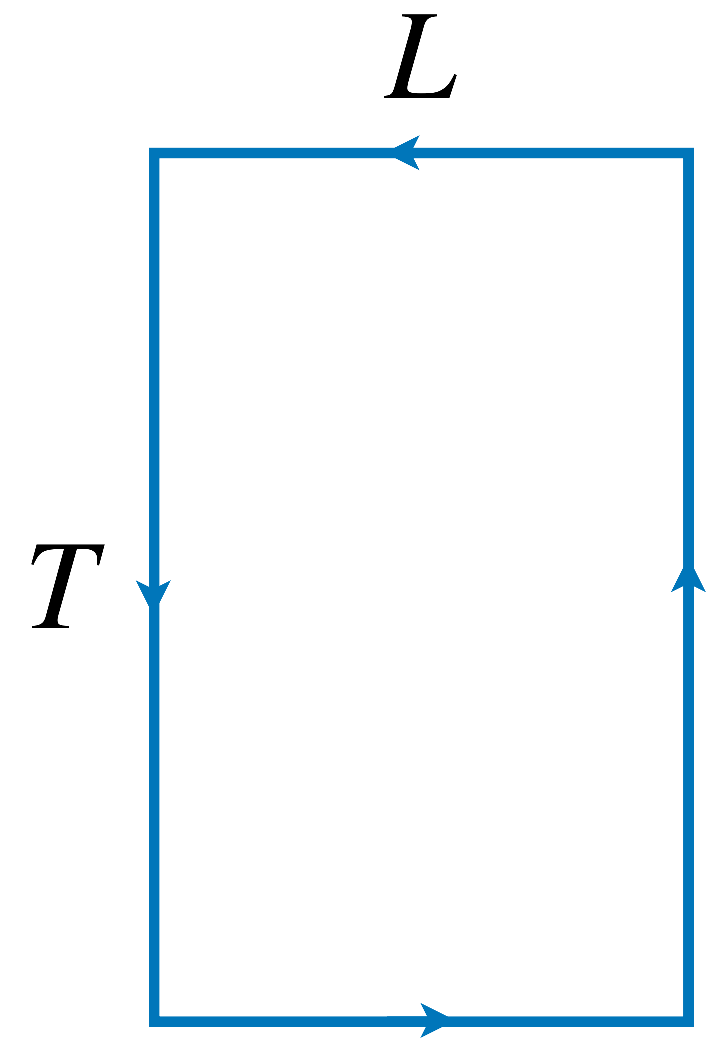

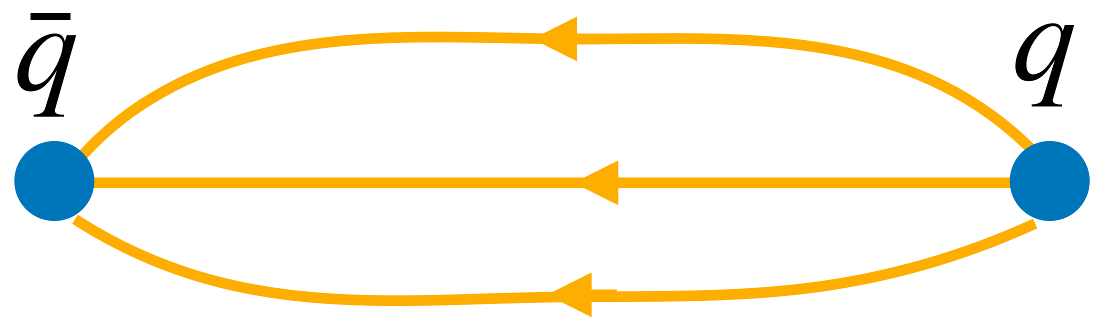

Effective QCD string

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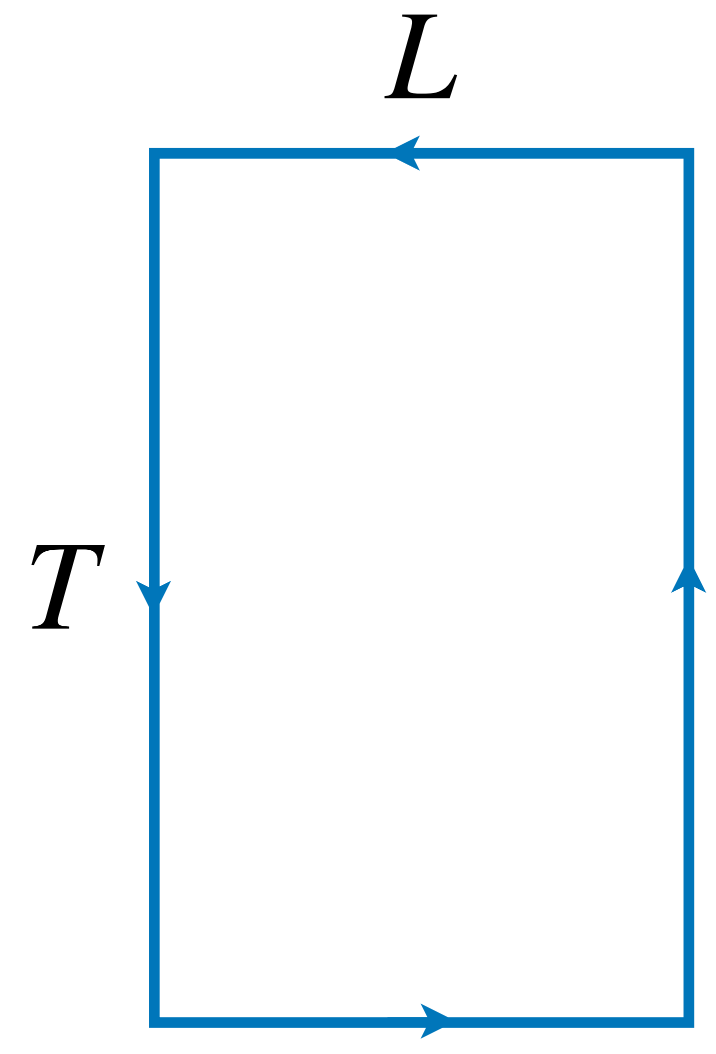

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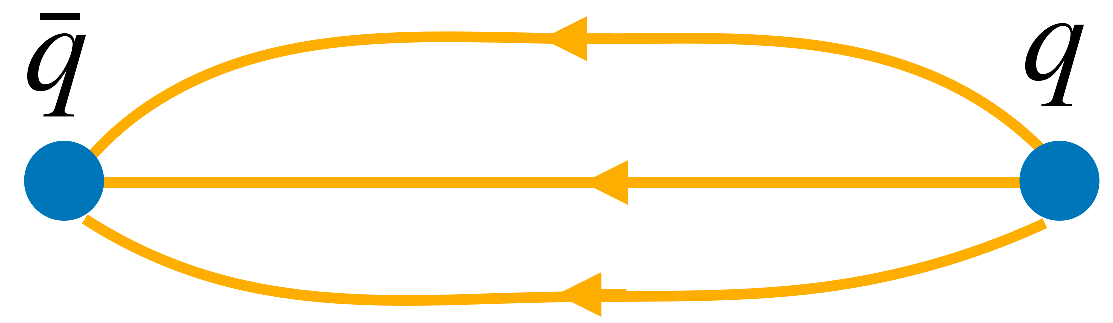
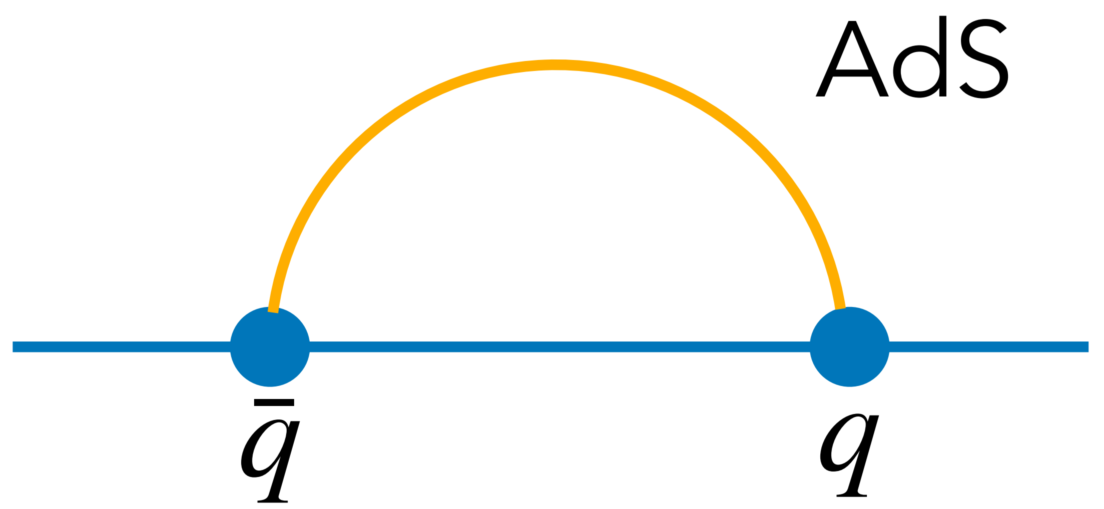
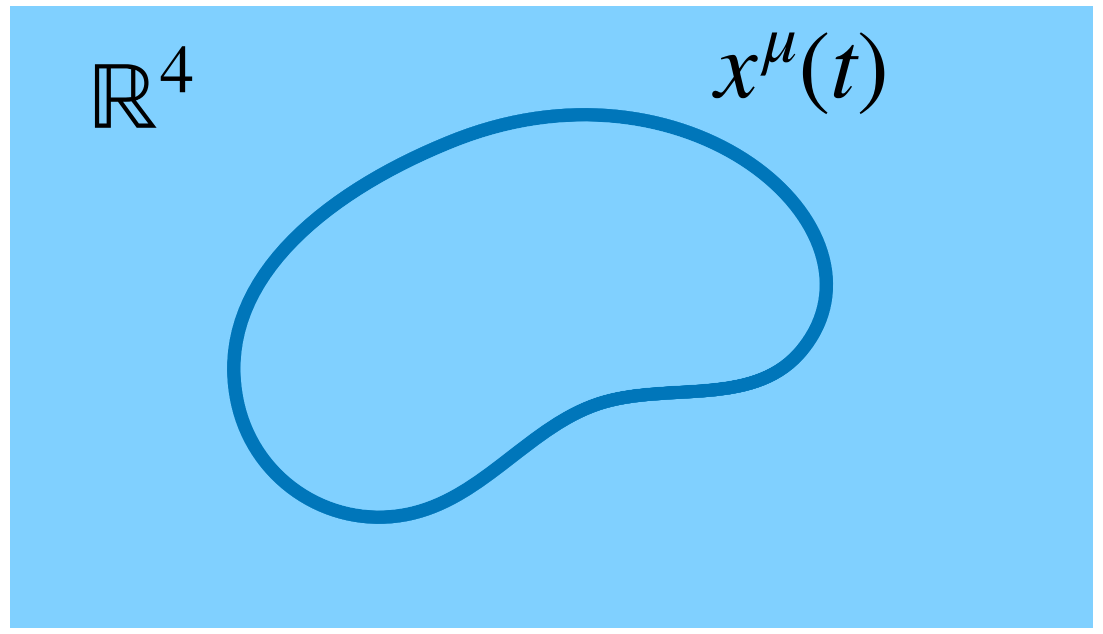


The area law is realized in AdS/CFT holographically.



Wilson loops in $\mathcal{N} = 4$ Super Yang-Mills



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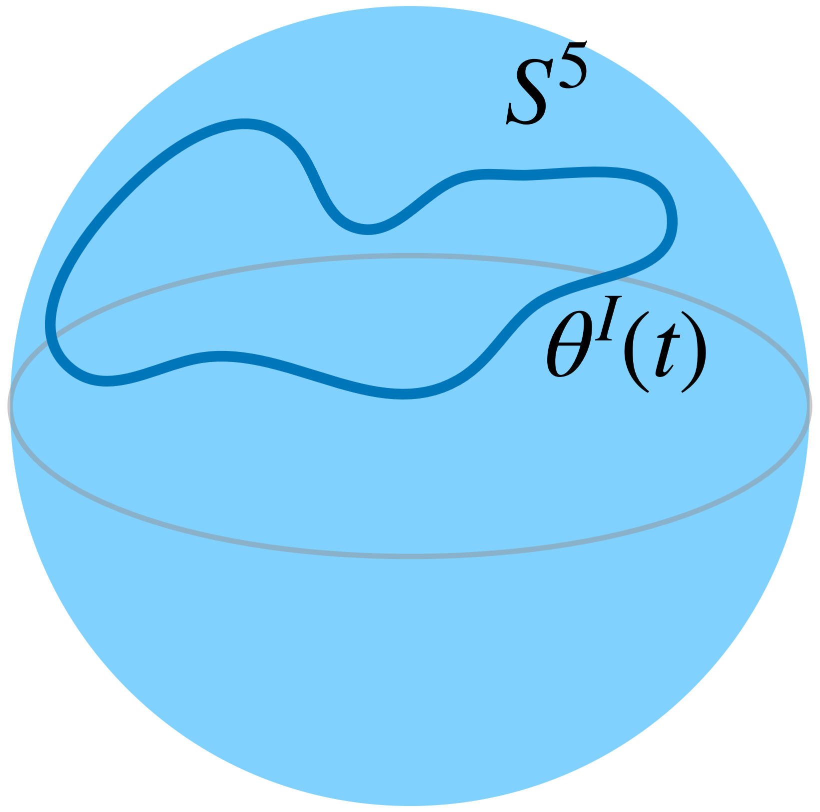

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Amenable to exact computations.

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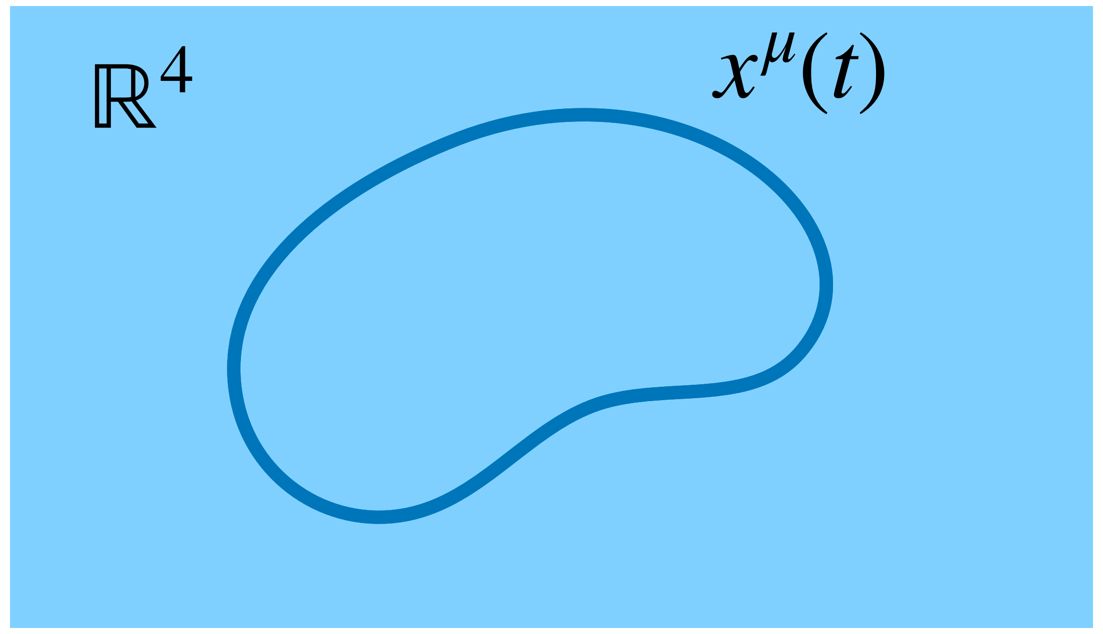

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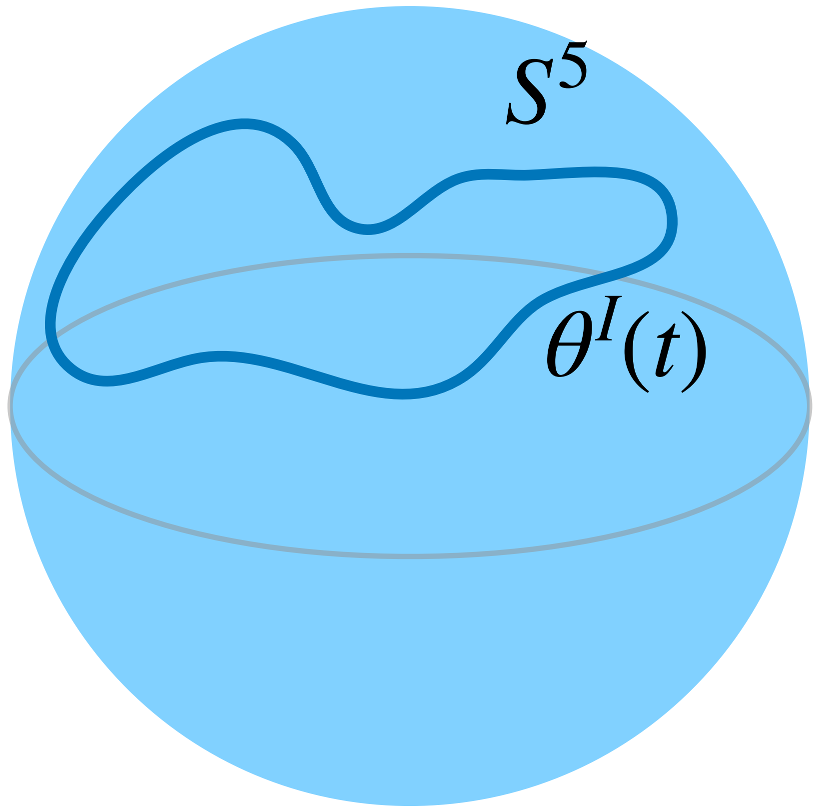

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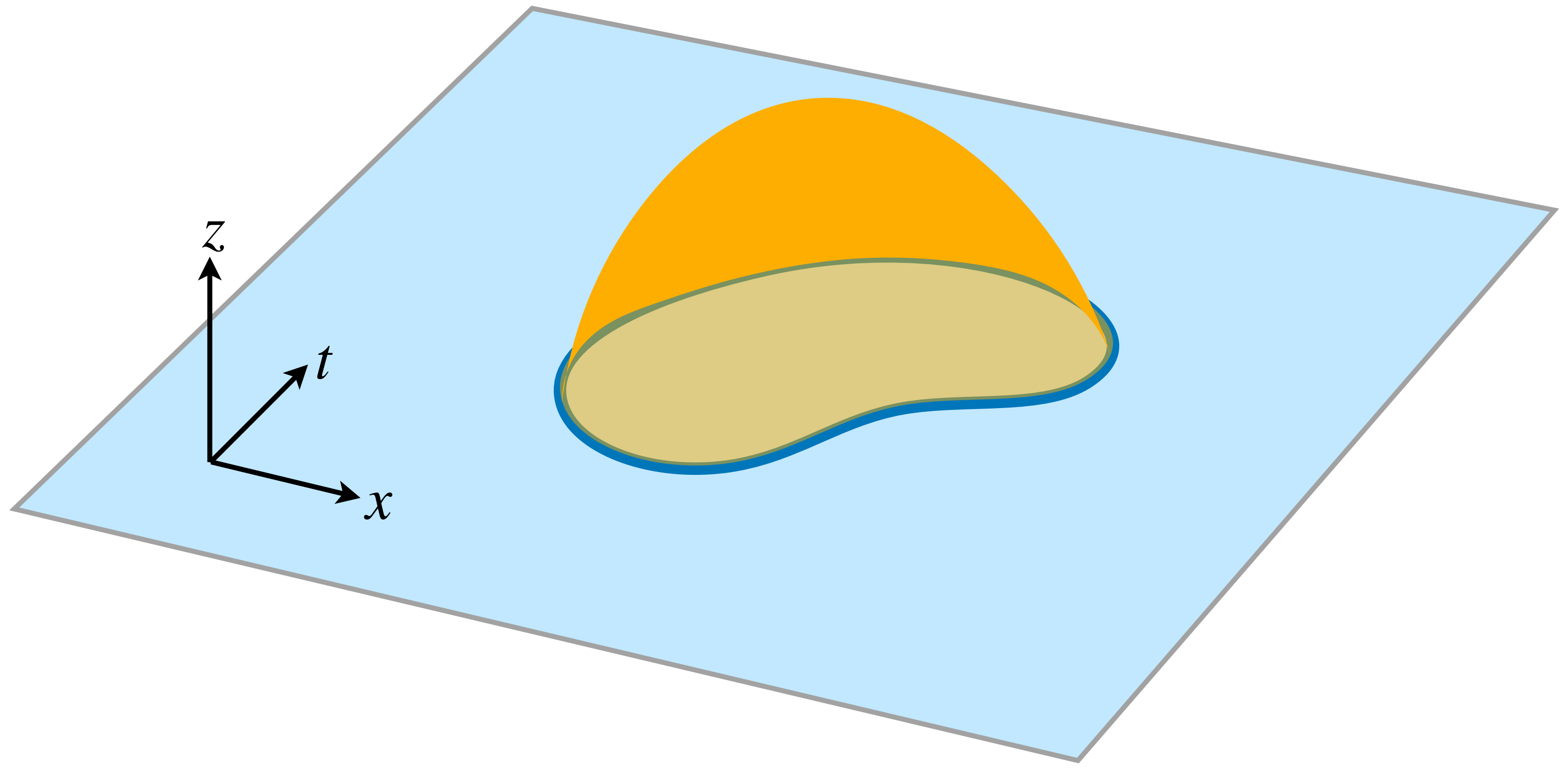
Dual to the open string in $AdS_5 \times S^5$ incident on the contour on the boundary: [Maldacena '98; Rey, Yee' 98]

$$\langle \mathcal{W} \rangle = Z_{\text{string}} \approx e^{-S_{\text{cl}}}$$



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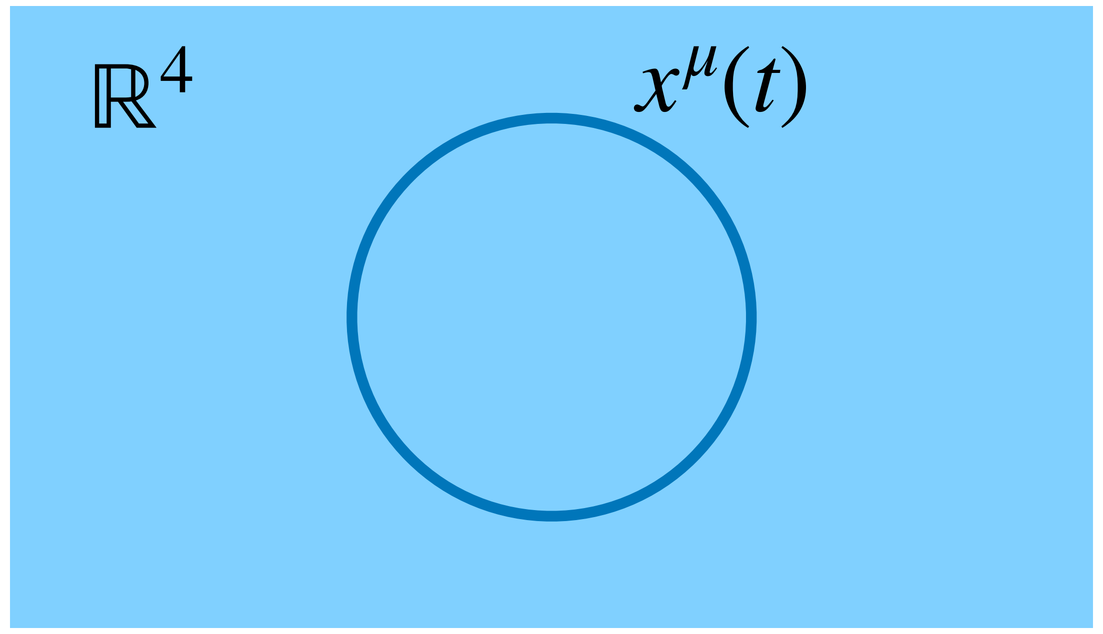
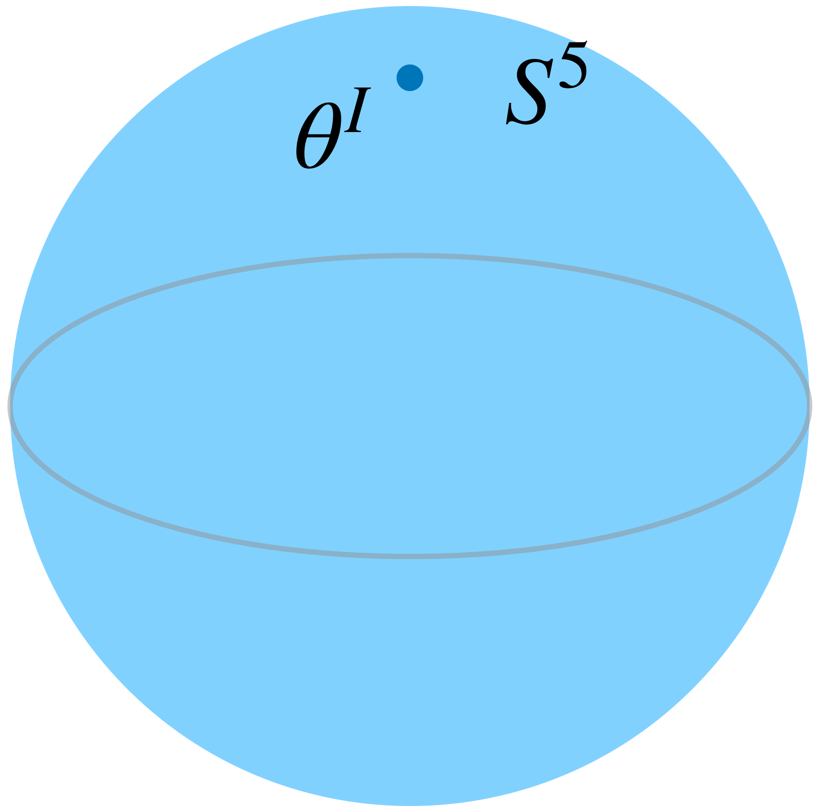


Circular spacetime contour: $x^\mu(t) = R(\cos(t), \sin(t), 0, 0)$. Point on S^5: $\theta^I = \delta^{I6}$.



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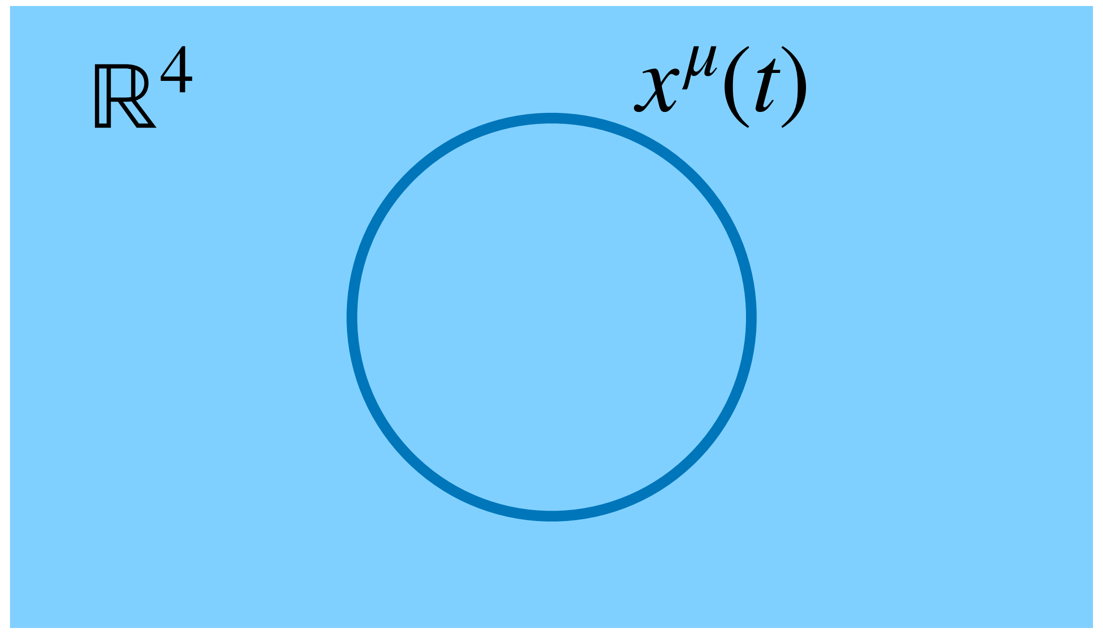

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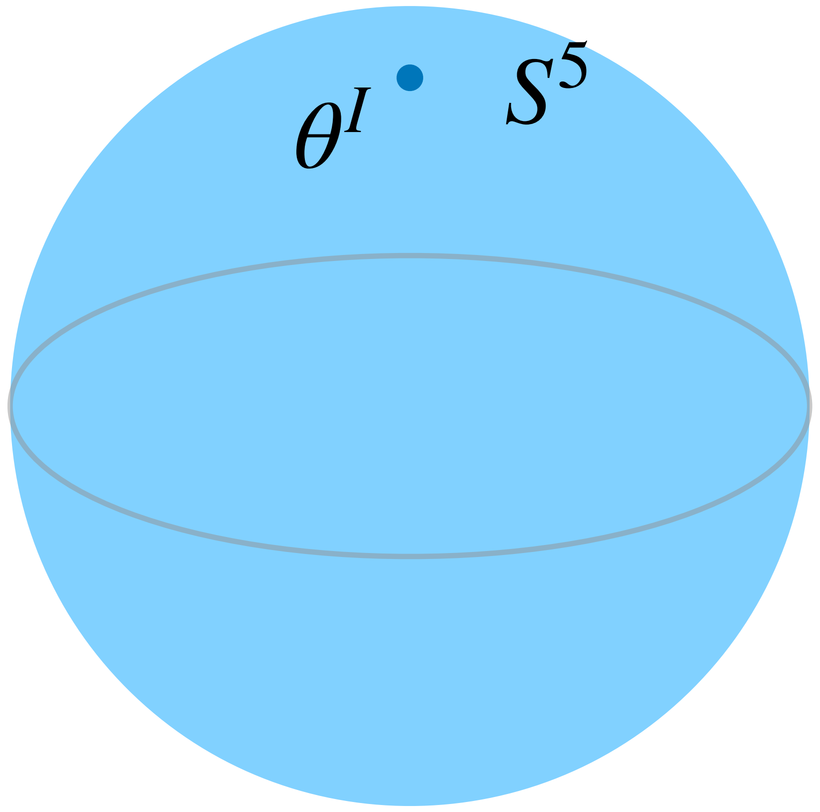

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[Erickson, Semenoff, Zarembo '00; Drukker, Gross '01; Pestun '07]

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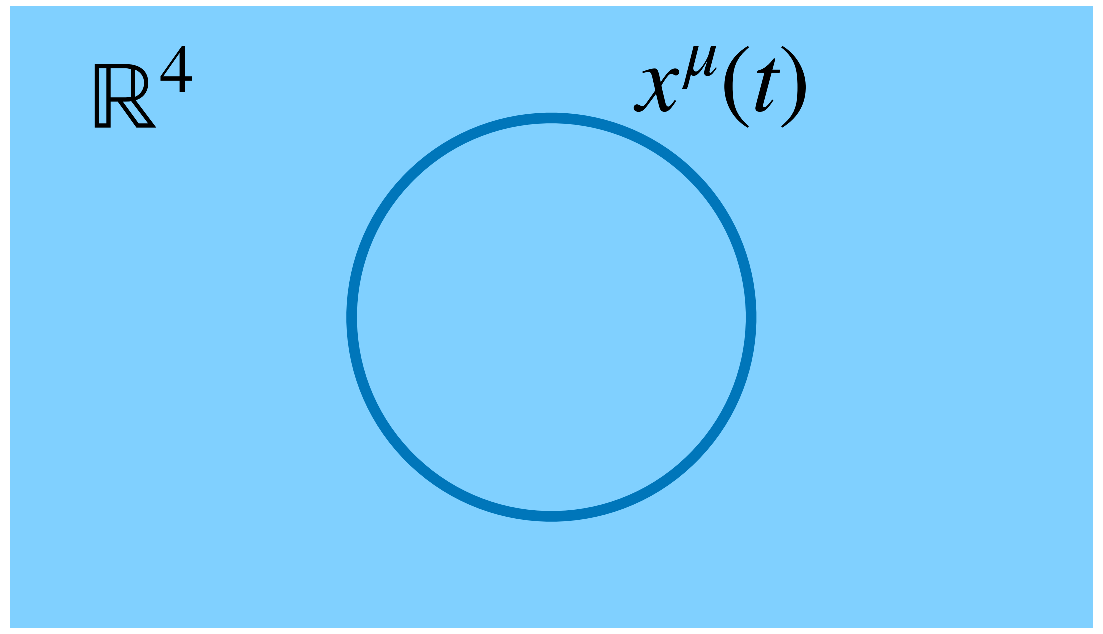

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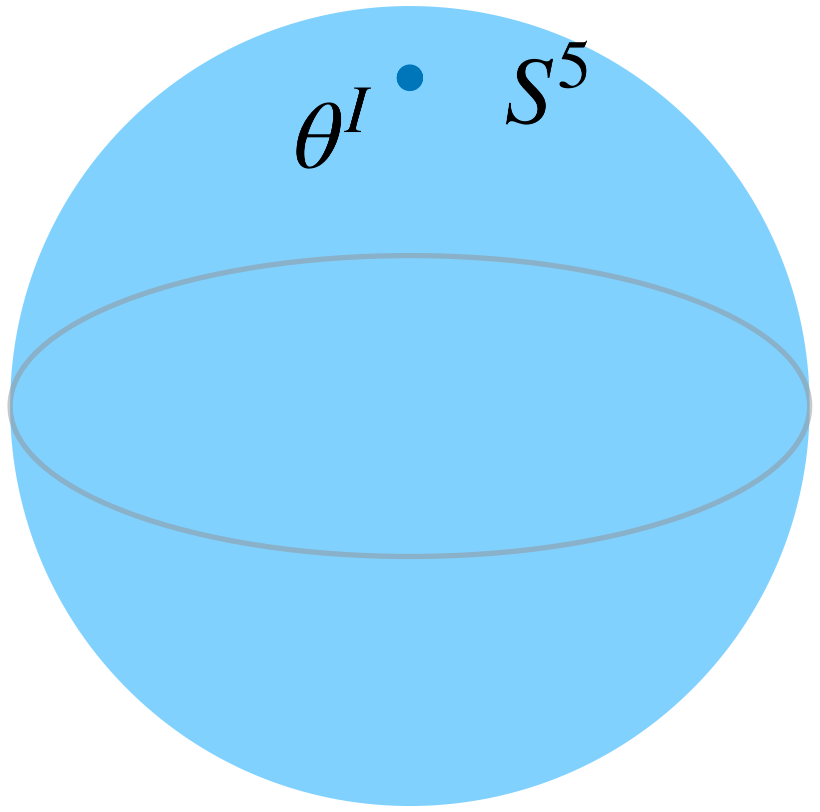

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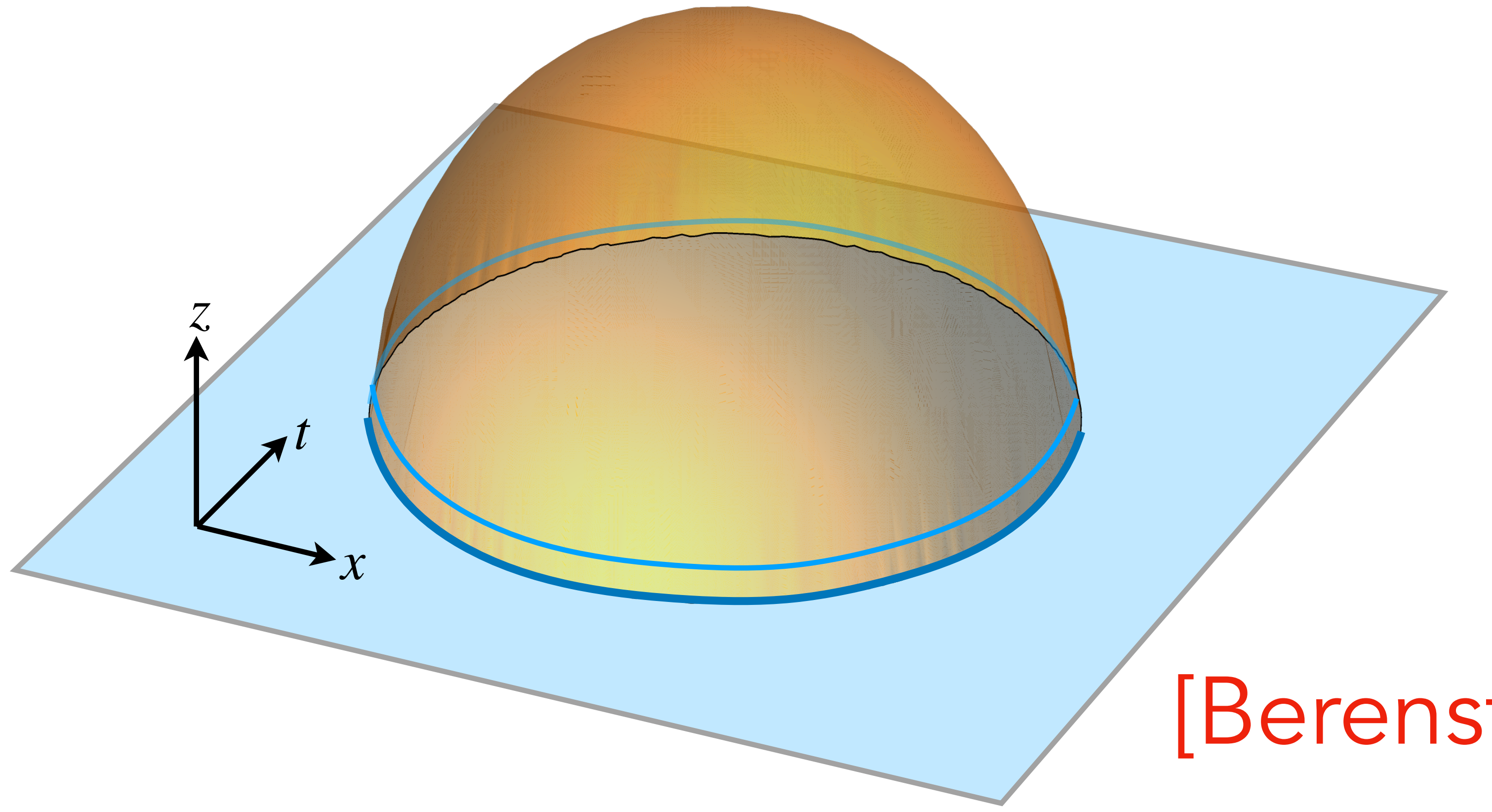

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[Berenstein, Corrado, Fischler, Maldacena 98; Drukker, Gross, Ooguri '99]

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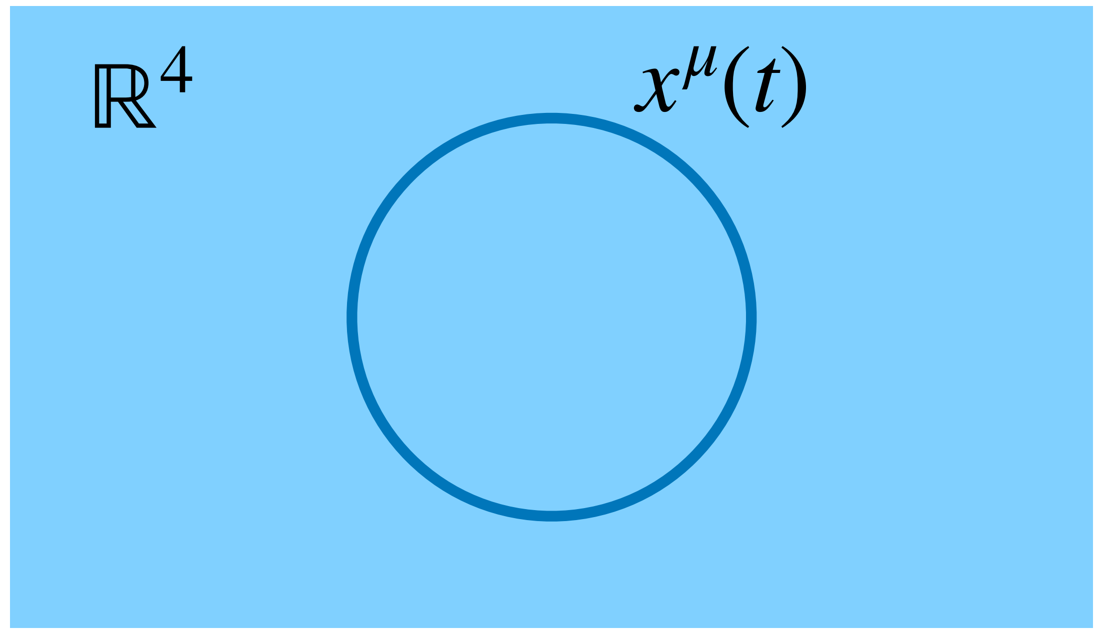

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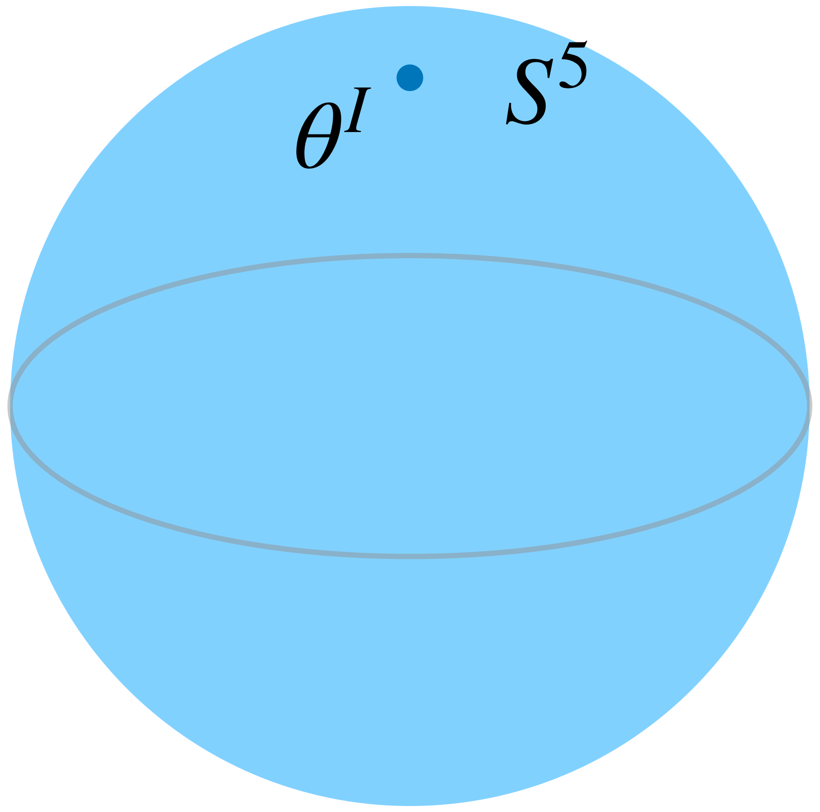

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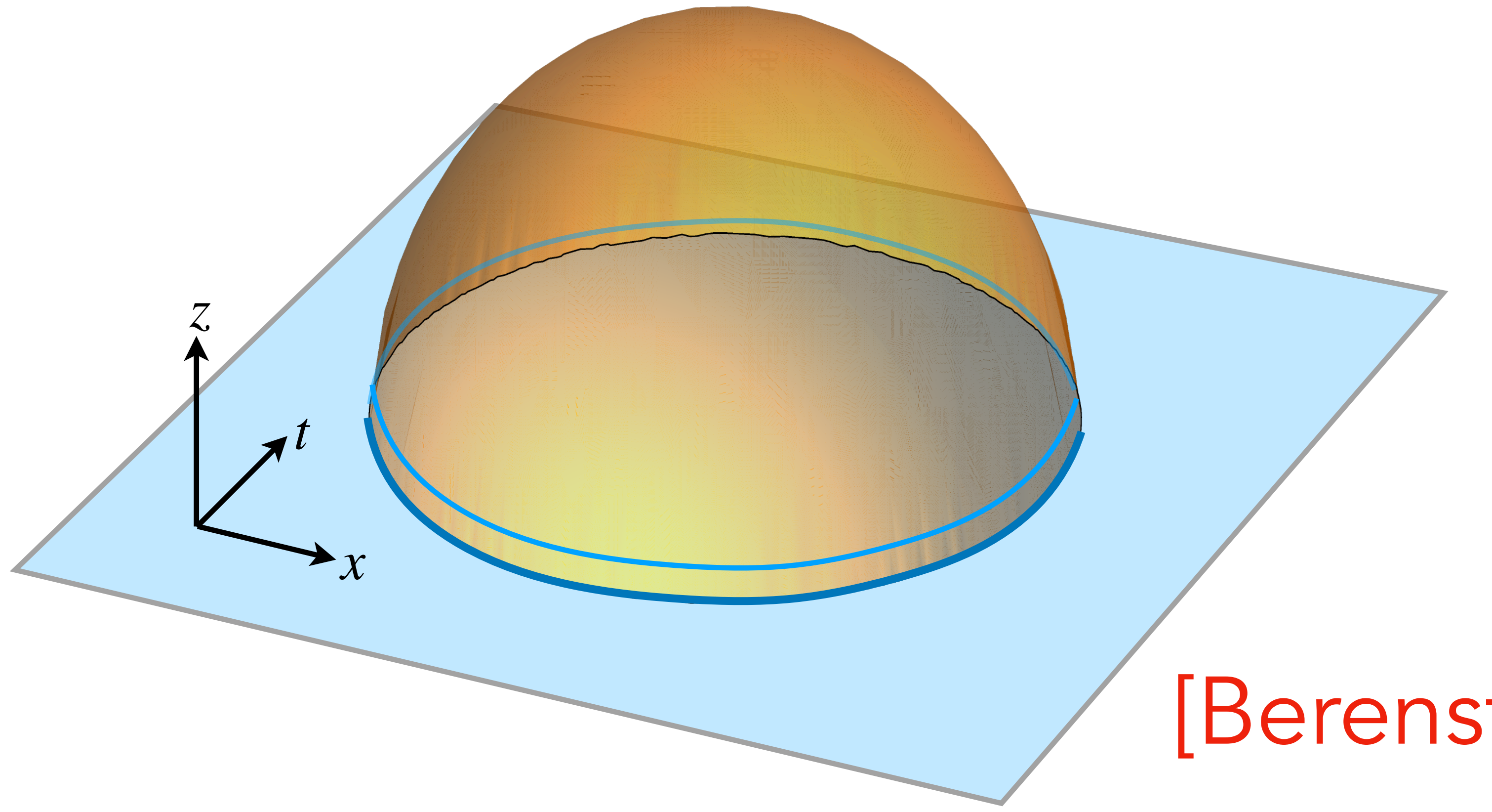

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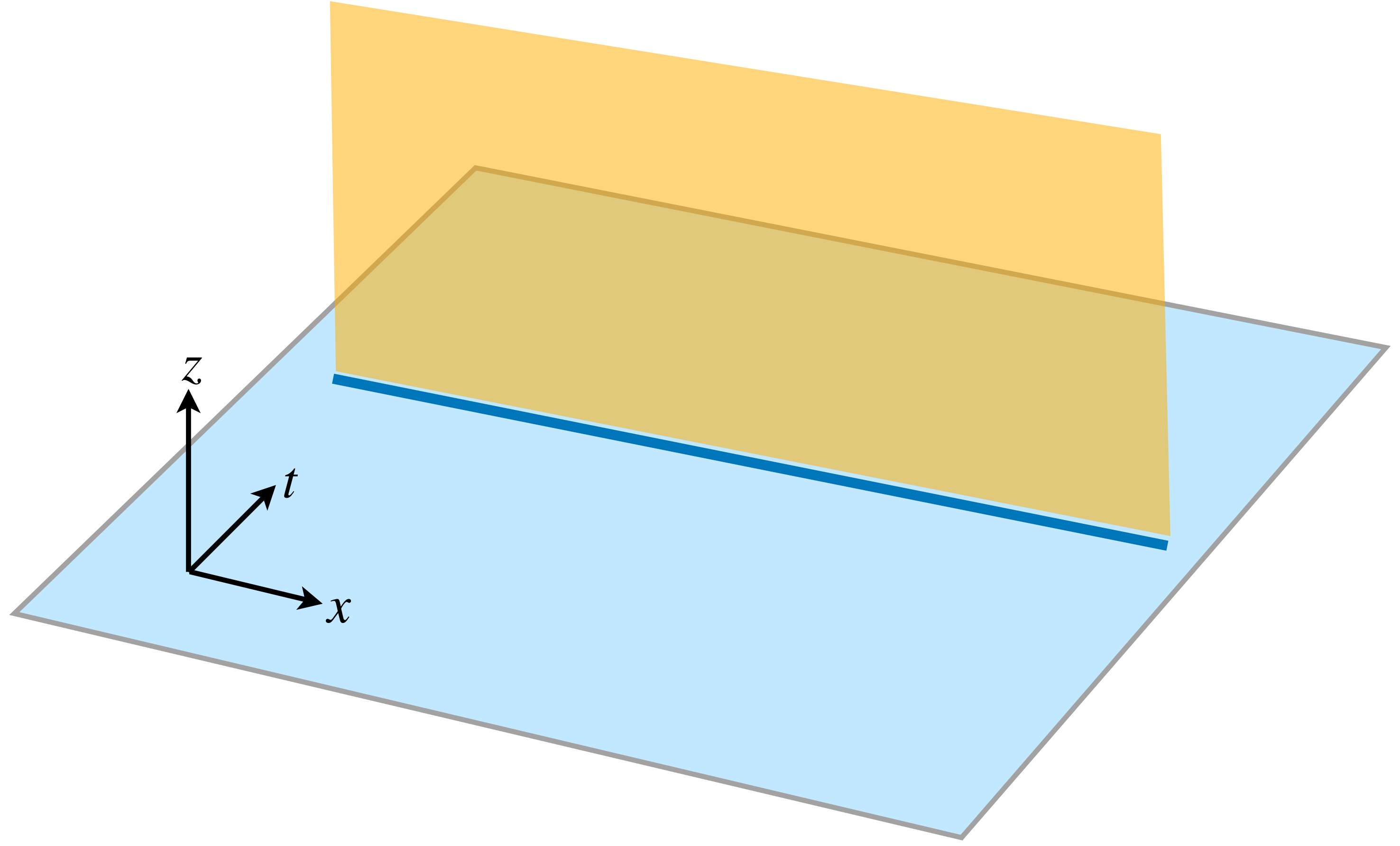

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Conformal defects

Defects in QFT have been studied a lot in recent years for a variety of reasons:

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In d-dim. CFT, one can study p-dim. *conformal defects*, which preserve $SO(p+1,1) \subset SO(d+1,1)$

One can then study the interplay of operators in the bulk and on the defect.

Wilson line defect CFT

The Wilson line (or circle) defines a 1d conformal defect. It breaks $PSU(2,2|4) \ni SO(5,1) \times SO(6)$ to the 1d superconformal group. $OSp(4^*|4) \ni SL(2,\mathbb{R}) \times SO(3) \times SO(5)$.

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We can define correlators of operators on the Wilson line

$$\langle O(x_1)\ldots O(x_n)\rangle \equiv \frac{1}{\langle \mathscr{W} \rangle}\left\langle \mathsf{P}[O(t_1)O(t_2)\ldots O(t_n)e^{\int_{-\infty}^{\infty} dt(iA_0+\Phi_6)}]\right\rangle_{\mathcal{N}=4\ \mathrm{SYM}}$$

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Defect correlators arise naturally when considering small fluctuations of the contour. [Polyakov, Rychkov '00; Drukker, Kawamoto '06, …]:

$$\langle \mathbb{D}_i(x_1)\Phi^I(x_2)\ldots\rangle = \frac{1}{\langle \mathscr{W} \rangle}\frac{\delta}{\delta x^i(x_1)}\frac{\delta}{\delta \theta^I(x_2)}\ldots\langle \mathscr{W}[x^i,\theta^I]\rangle = \frac{1}{Z_{\mathrm{string}}}\frac{\delta}{\delta x^i(x_1)}\frac{\delta}{\delta \theta^I(x_2)}\ldots Z_{\mathrm{string}}[x^i,\theta^I]$$

Where $\mathbb{D}_i = iF_{0i} + D_i\Phi^6$, $i = 1,2,3$, is the displacement operator and $\Phi^I(t)$, $I = 1,\ldots,5$, are the orthogonal scalars.

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These are dual to the 3 transverse fluctuations of the string in AdS_5 and the 5 transverse fluctuations in S^5

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The $SL(2,\mathbb{R})$ fixes the two and three-point functions:

$$\langle O_i(x_1) O_j(x_2) \rangle = \frac{\mathcal{N}_i \delta_{ij}}{x_{12}^{2\Delta_i}}, \qquad \langle O_i(x_1) O_j(x_2) O_k(x_3) \rangle = \frac{C_{ijk}}{|x_{12}|^{\Delta_{ij|k}} |x_{13}|^{\Delta_{ik|j}} |x_{23}|^{\Delta_{jk|i}}},$$

Four point functions:

$$\langle O(x_1) \dots O(x_4) \rangle = \frac{1}{x_{12}^{2\Delta} x_{34}^{2\Delta}} G(\chi),$$

Where $\chi = \dfrac{x_{12} x_{34}}{x_{13} x_{24}}$.

Wilson line defect CFT

This defect CFT has been studied extensively:

- At weak coupling using perturbation theory [Cooke, Dekel, Drukker '17; Komatsu, Kiryu '18; Barrat, Liendo, Peveri, Plefka '21 …]

- At strong coupling by considering fluctuations of the dual string [Giombi, Roiban, Tseytlin '17]

- Using localization [Correa, Henn, Maldacena, Sever '12; Giombi, Komatsu '18]

- Using integrability [Cavaglia, Gromov, Julius, Preti '21 '22, …]

- Using the analytic conformal bootstrap [Liendo, Meneghelli, Mitev '18; Ferrero, Meneghelli '21]

Correlators on the string

Our focus will be on the four-point functions at strong coupling in the planar limit (i.e. $N \to \infty, \lambda = g_{YM}^2 N$: fixed, and $\lambda \gg 1$) involving a single scalar

In the dual string, this corresponds to considering weakly interacting fluctuations of the string (i.e. $g_s = 0$, $T_s = \ell_s^{-1} = \frac{\sqrt{\lambda}}{2\pi} \gg 1$) in a subspace $\mathrm{AdS}_2 \times S^1 \subset \mathrm{AdS}_5 \times S^5$.

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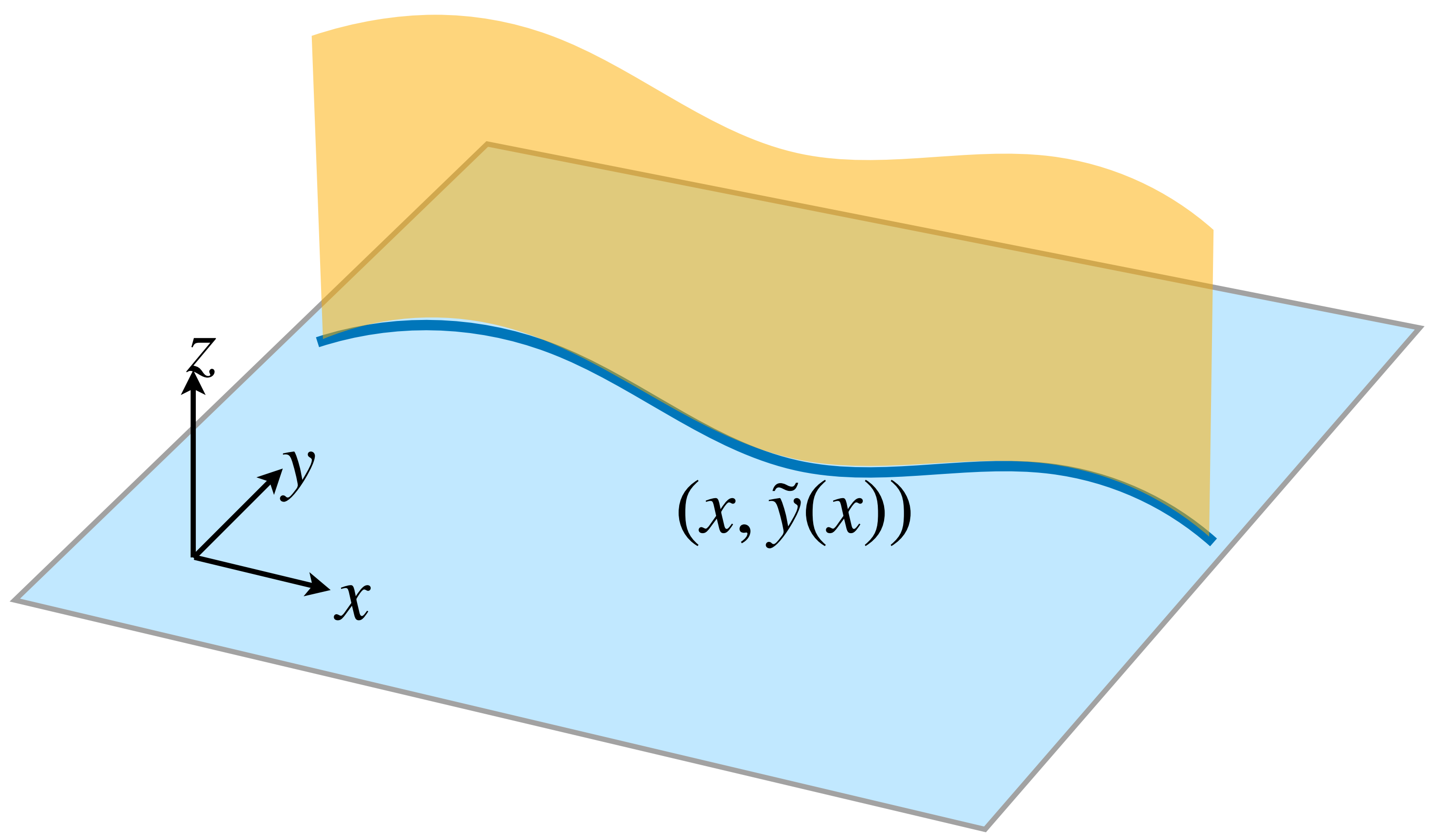
We work with Poincare coordinates $x \in \mathbb{R}$, $z \in [0,\infty)$ on AdS_2 and polar coordinate $y \in [0,2\pi]$ on S^1:

$$ds^2 = \frac{dz^2 + dx^2}{z^2} + dy^2.$$



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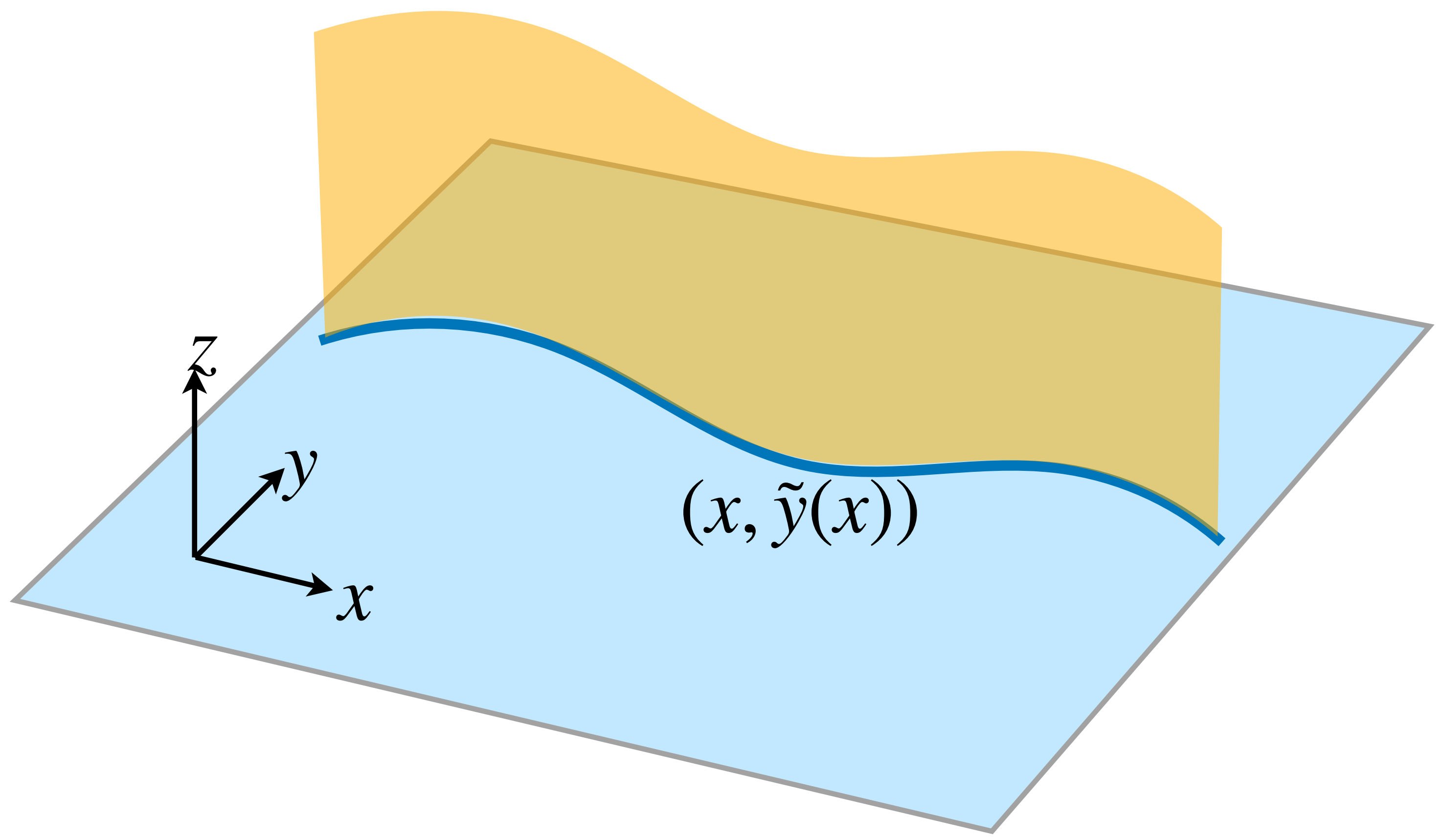

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The Wilson loop / string is then a function of $\tilde{y}(x)$ only and the correlators are:

$$\langle \Phi(x_1)\ldots\Phi(x_4)\rangle = \langle y(x_1)\ldots y(x_4)\rangle = \frac{1}{Z_{\text{string}}}\frac{\delta}{\delta\tilde{y}(x_1)}\cdots\frac{\delta}{\delta\tilde{y}(x_4)} Z_{\text{string}}[\tilde{y}]\Big|_{\tilde{y}=0.} \approx e^{S_{\text{cl}}[\tilde{y}]}\frac{\delta}{\delta\tilde{y}(x_1)}\cdots\frac{\delta}{\delta\tilde{y}(x_4)} e^{-S_{\text{cl}}[\tilde{y}]}\Big|_{\tilde{y}=0}.$$

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With boundary condition: $y(0,t) = \tilde{y}(t)$. The induced metric is $g_{\alpha\beta} = \frac{1}{s^2}\delta_{\alpha\beta}$.

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We can expand in small fluctuations:

$$S = T_s \int d^2\sigma \sqrt{g}\,\left(L_0 + L_2 + L_4 + \ldots\right),$$

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Where:

$$L_0 = 1, \qquad L_2 = \frac{1}{2} g^{\alpha\beta}\partial_\alpha y \partial_\beta y, \qquad L_4 = -\frac{1}{8}(g^{\alpha\beta}\partial_\alpha y \partial_\beta y)^2 \,.$$

We see that $y(s,t)$ is a massless field in AdS_2 with a tower of interactions. Recall that $m^2 = \Delta(\Delta-1)$.

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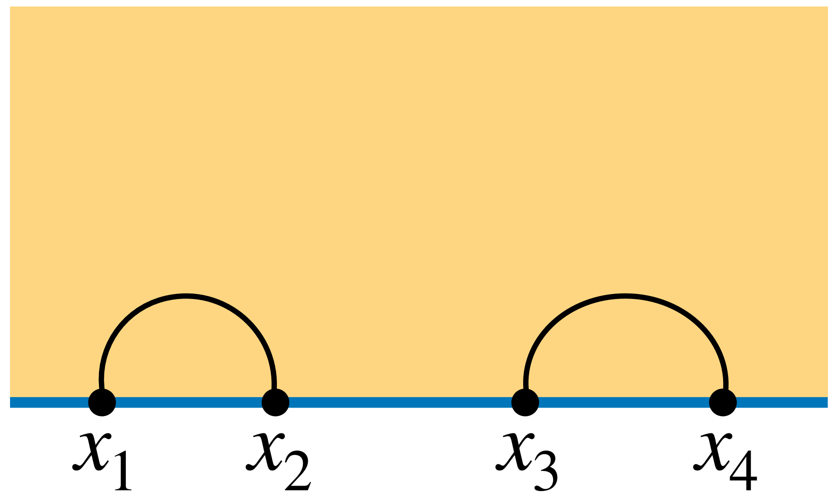

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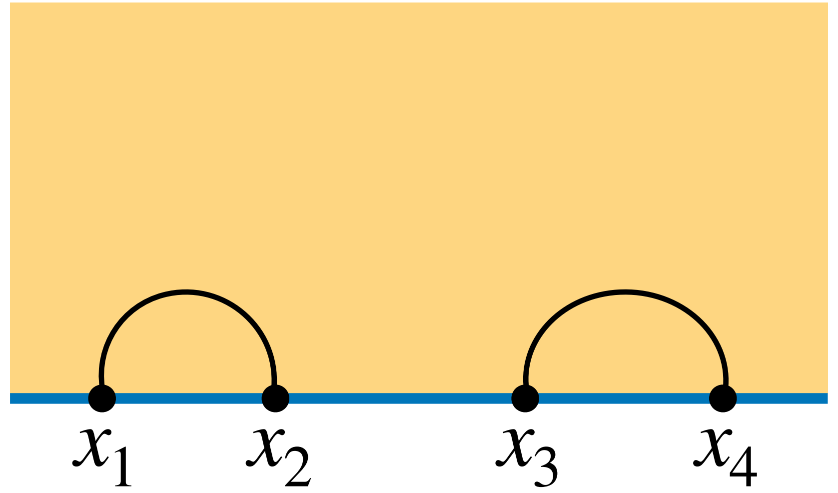

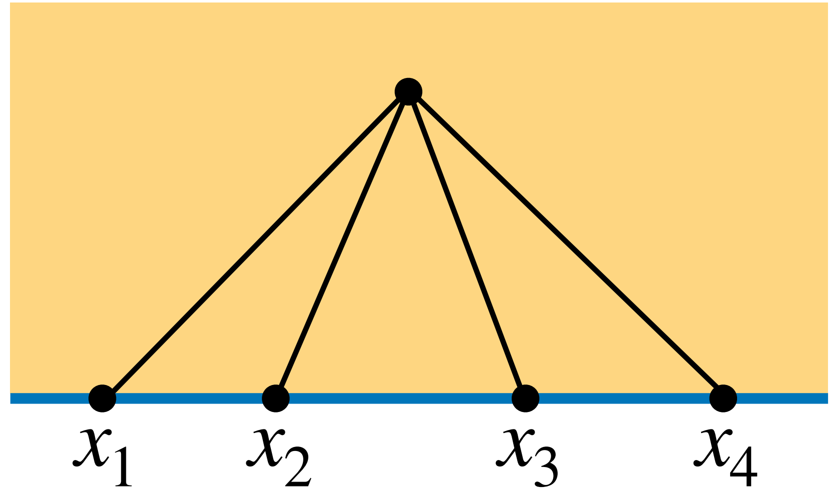

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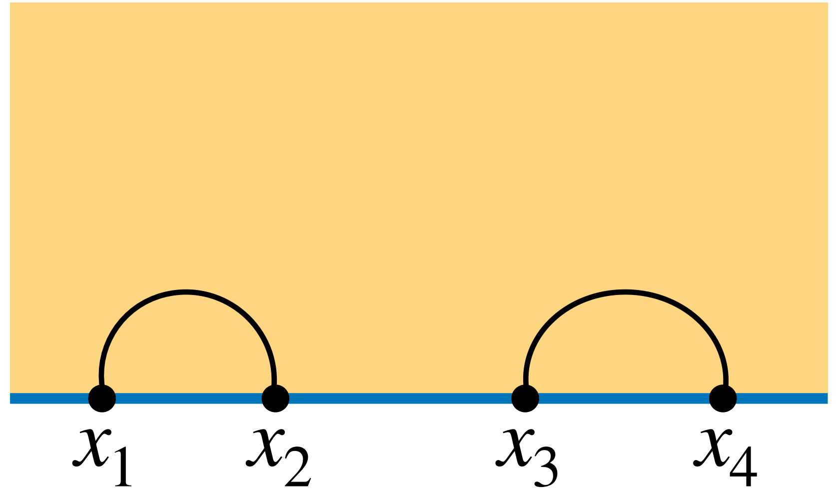
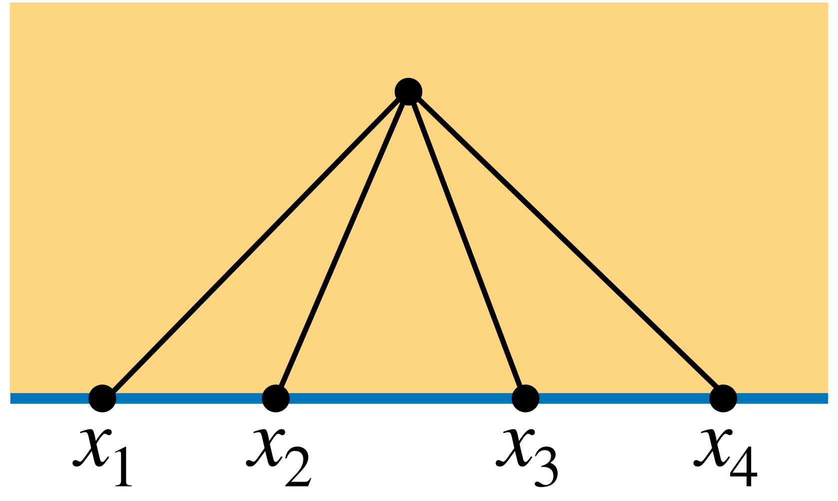
[Giombi, Roiban, Tseytlin '17]





$$\langle y_1 y_2 y_3 y_4 \rangle = \left[\frac{T_s^2}{\pi^2} \frac{1}{x_{12}^2 x_{34}^2} + \text{perms} \right] + \left[T_s \int d^2\sigma \sqrt{g}\, g^{\alpha\beta} \partial_\alpha K(s,t,x_1) \partial_\beta K(s,t,x_2) g^{\gamma\delta} \partial_\gamma K(s,t,x_3) \partial_\delta K(s,t,x_4) + \text{perms} \right] + \ldots$$

We find the normalized four-point function takes the form:

$$\frac{\langle y_1 y_2 y_3 y_4 \rangle}{\langle y_1 y_2 \rangle \langle y_3 y_4 \rangle} = G_{\text{free}}(\chi) + \frac{1}{2\pi T_s} G_{\text{tree}}(\chi) + \ldots,$$

Where

$$G_{\text{free}}(\chi) = 1 + \chi^2 + \frac{\chi^2}{(1-\chi)^2},$$

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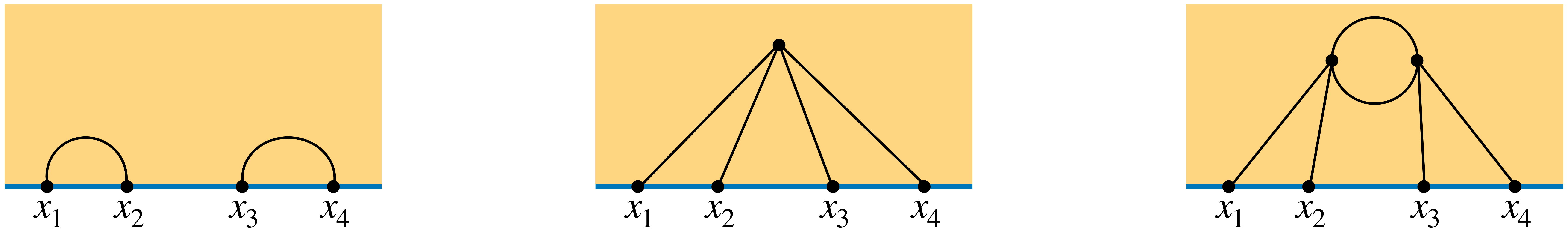

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Part II

OTOCs and scattering amplitudes on the AdS_2 string

Out-of-time-order correlators (OTOCs)

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The natural generalization in a quantum system is $\mathrm{Tr}\left(e^{-\beta H}[V, W(t)]^2\right)$. The exponential growth is driven by the out-of-time-order terms:

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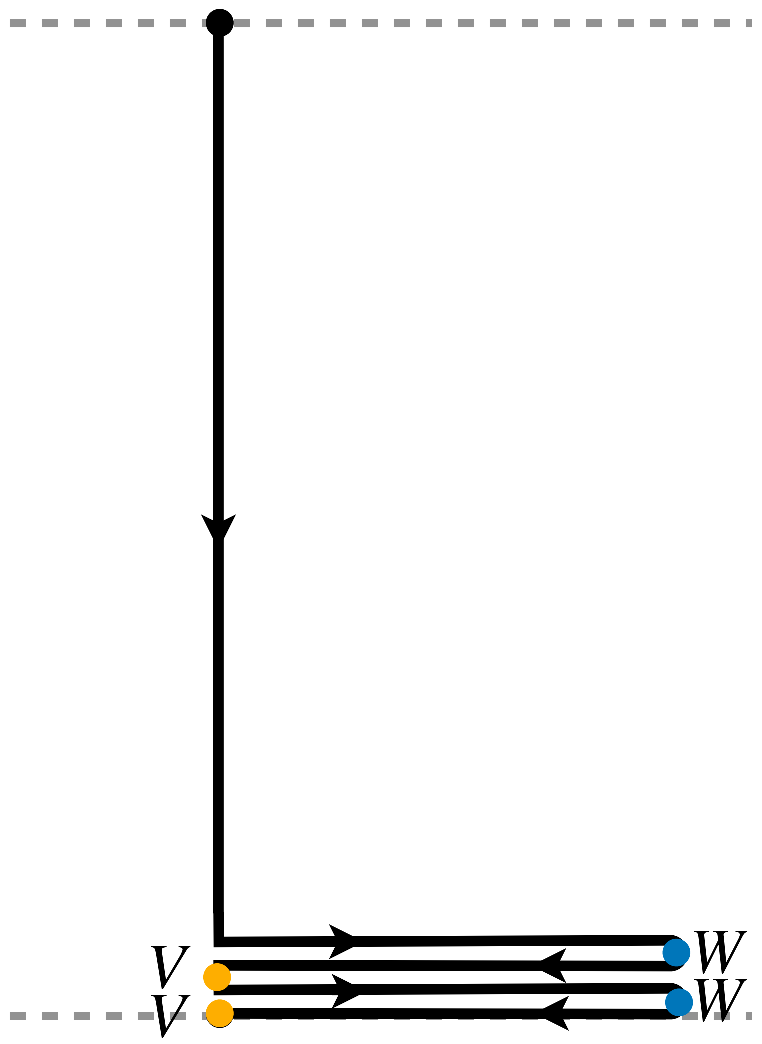

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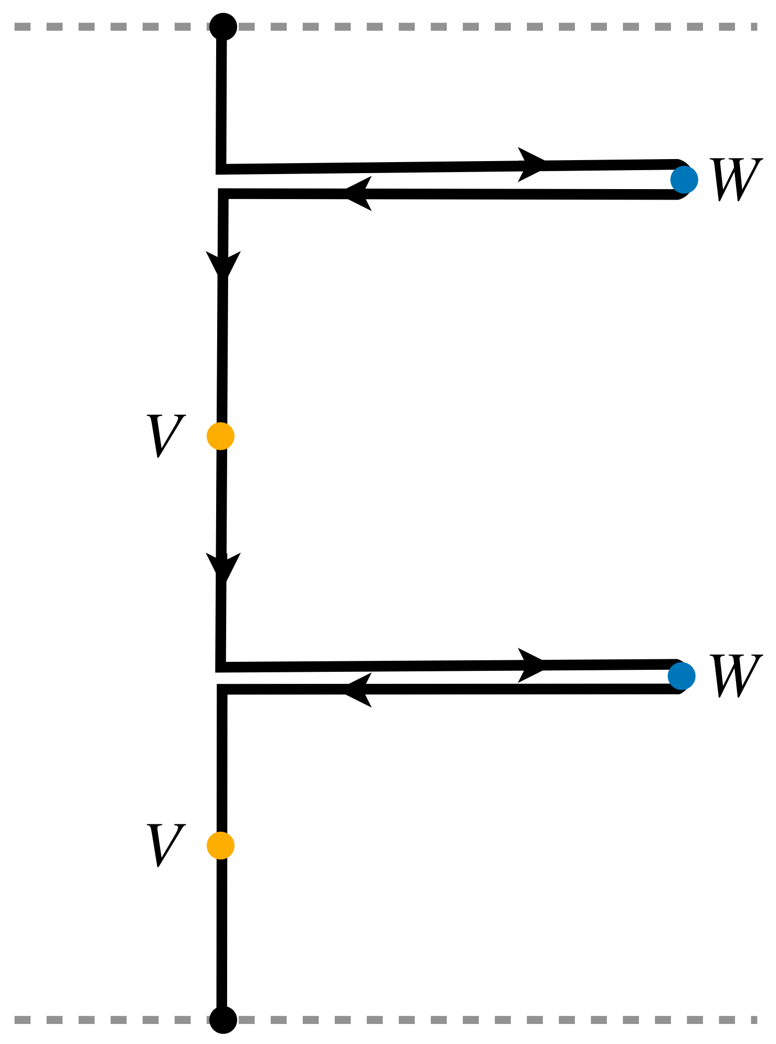

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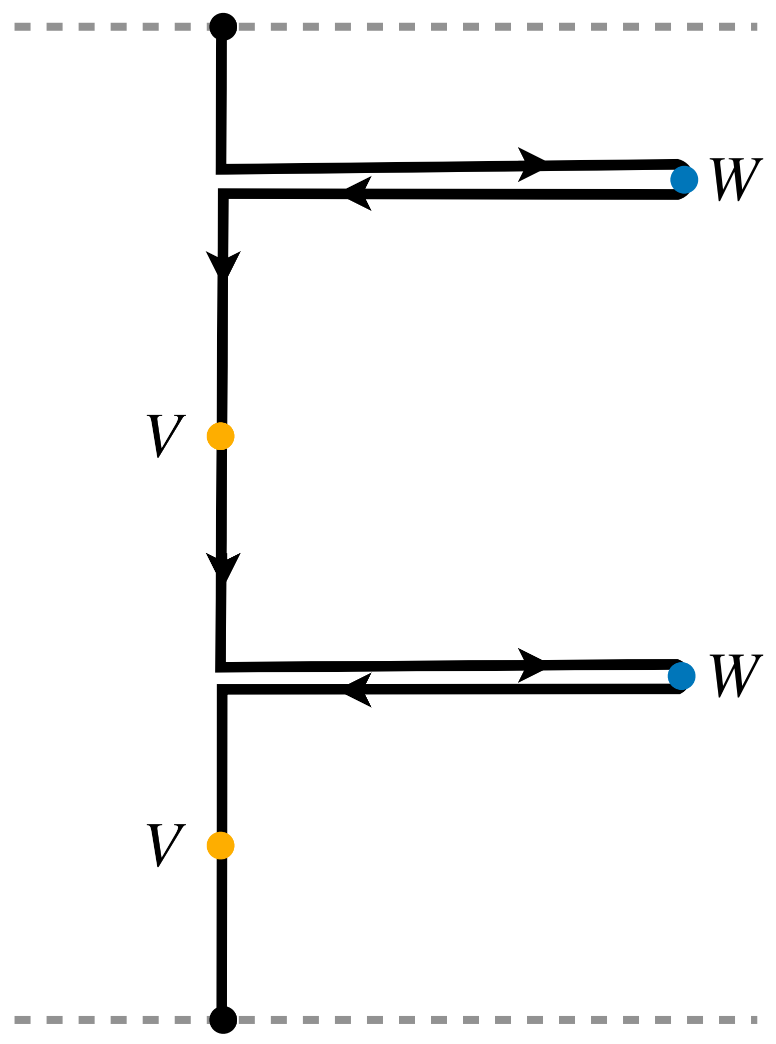

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The quantum Lyapunov exponent satisfies a bound [MSS '15]:

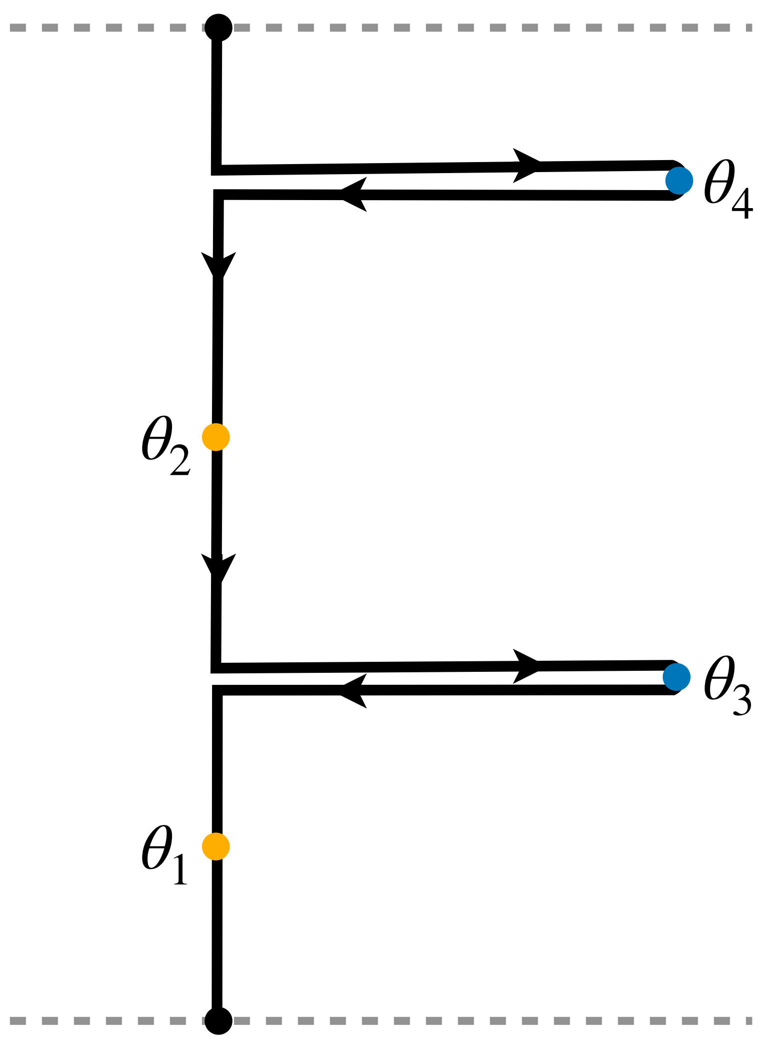
$$\lambda_L \leq \frac{2\pi}{\beta},$$

Einstein gravity saturates this bound [SS '14]. E.g., in $\mathcal{N} = 4$ SYM, $\langle VW(t)VW(t)\rangle_\beta = 1 - \frac{\#}{N^2} e^{\frac{2\pi}{\beta}t} = 1 - \# G_N e^{\frac{2\pi}{\beta}t}$.

OTOC on the string via static gauge

We were interested in studying OTOCs in the Wilson line dCFT. To compute the OTOC, we can analytically continue the euclidean four-point function. Let $x_i = \tan(\frac{\theta_i}{2})$ where (setting $\beta = 2\pi$):

$$\theta_1 = \frac{3\pi}{2}, \qquad \theta_2 = \frac{\pi}{2}, \qquad \theta_3 = \pi + it, \qquad \theta_4 = it.$$



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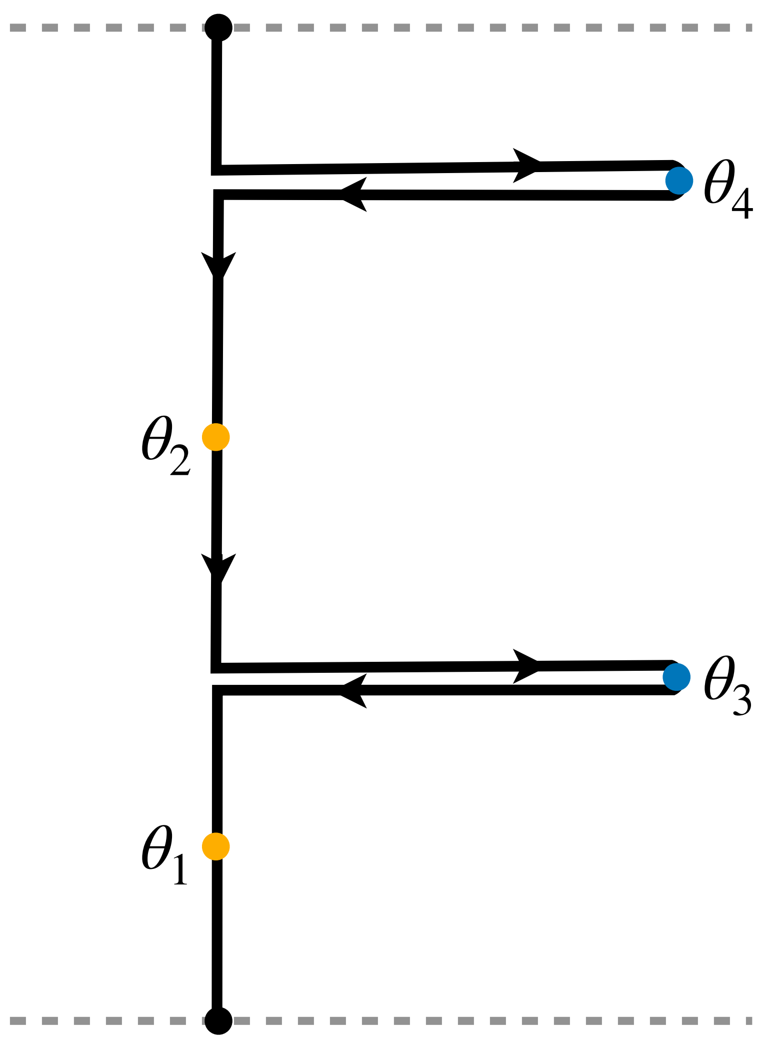

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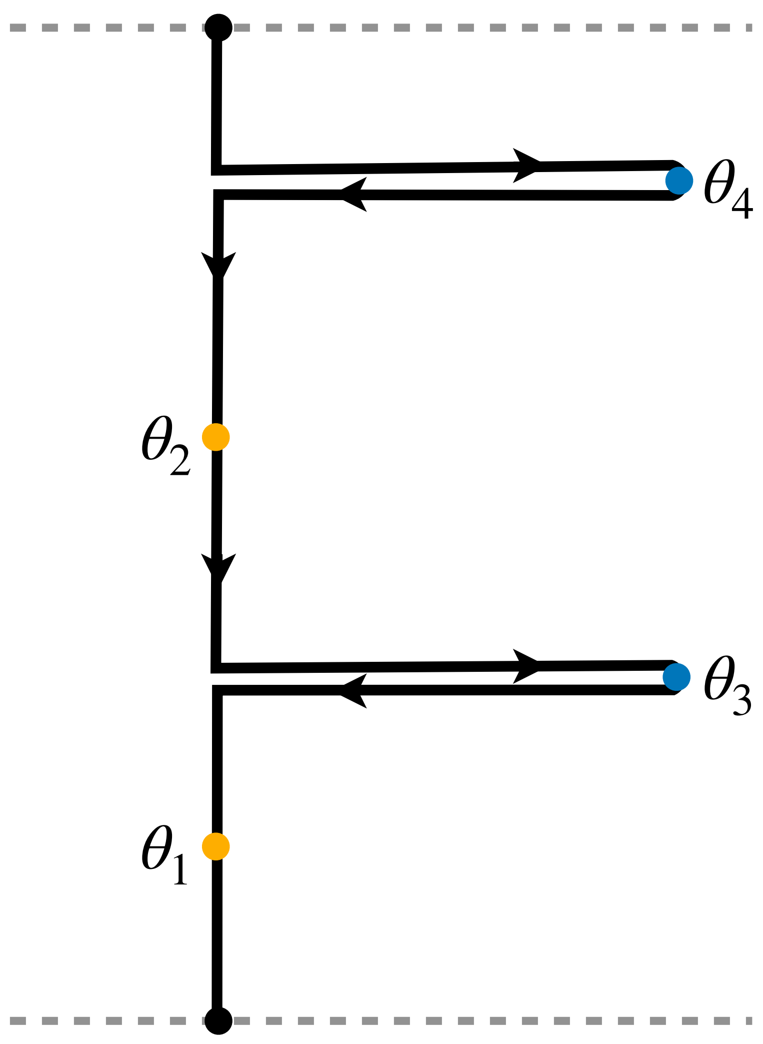

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Where $G_{\text{free}}(\chi) = 1 + \ldots,\quad G_{\text{tree}}(\chi) = \frac{-2+\ldots}{2\chi}\log((1-\chi)^2) + \ldots$. Letting $\chi(t) = \frac{2}{1 - i\sinh t}$:

$$\frac{\langle yy(t)yy(t) \rangle}{\langle yy \rangle \langle yy \rangle} = 1 - \frac{e^t}{4T_s} + \ldots$$

This saturates the chaos bound: $\lambda_L = \frac{2\pi}{\beta}$.

[Maldacena, Stanford, Yang '17; Murata '17; de Boer, Llabres, Pedraza, Vegh '17]

OTOC up to three-loops

[Ferrero, Meneghelli '21]

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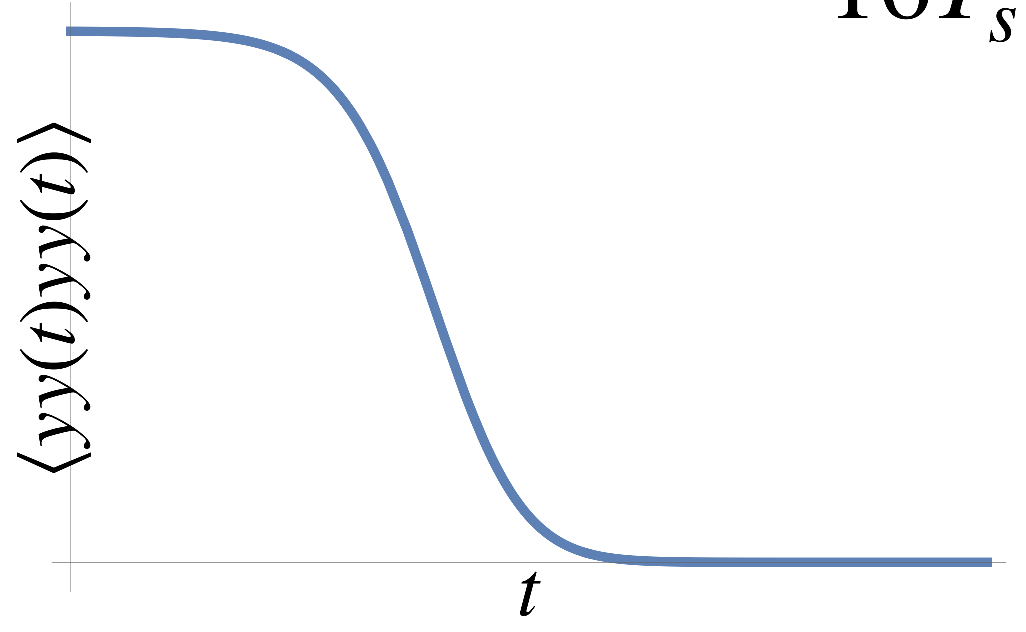
$$= \frac{1}{\kappa^2} U(2,1,\kappa^{-1})$$

taking $t, T_s \to \infty$ with $\kappa = \dfrac{e^t}{16T_s}$ fixed



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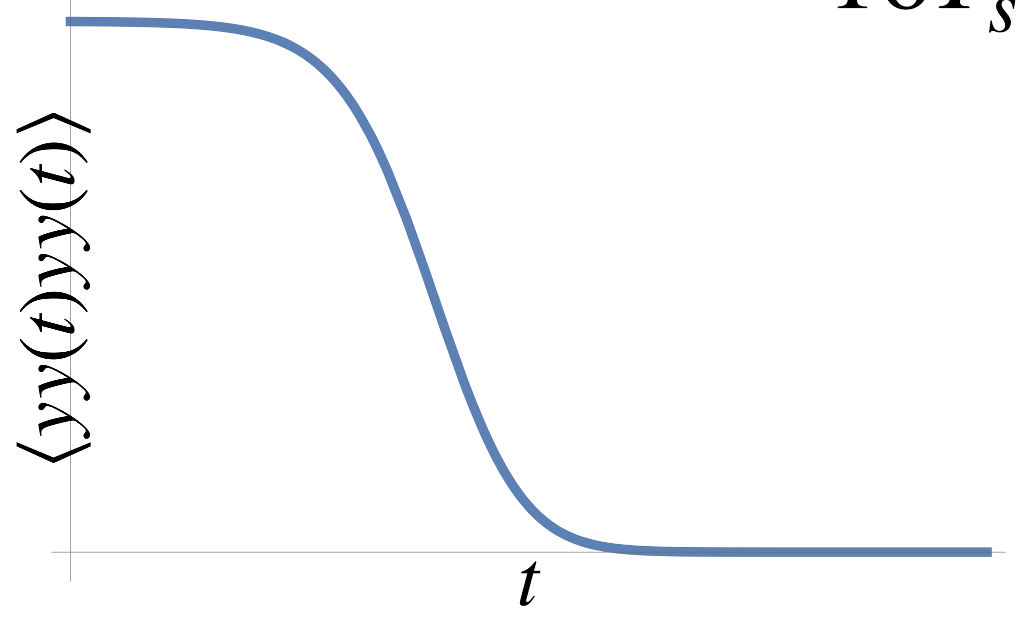

This is the same result as in JT gravity(!)

[Maldacena, Stanford, Yang '16; Lam, Mertens, Turiaci, Verlinde '18]

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[Banerjee, Kundu, Poojary '18; Vegh, 19; Gutiez, Hoyos '22]

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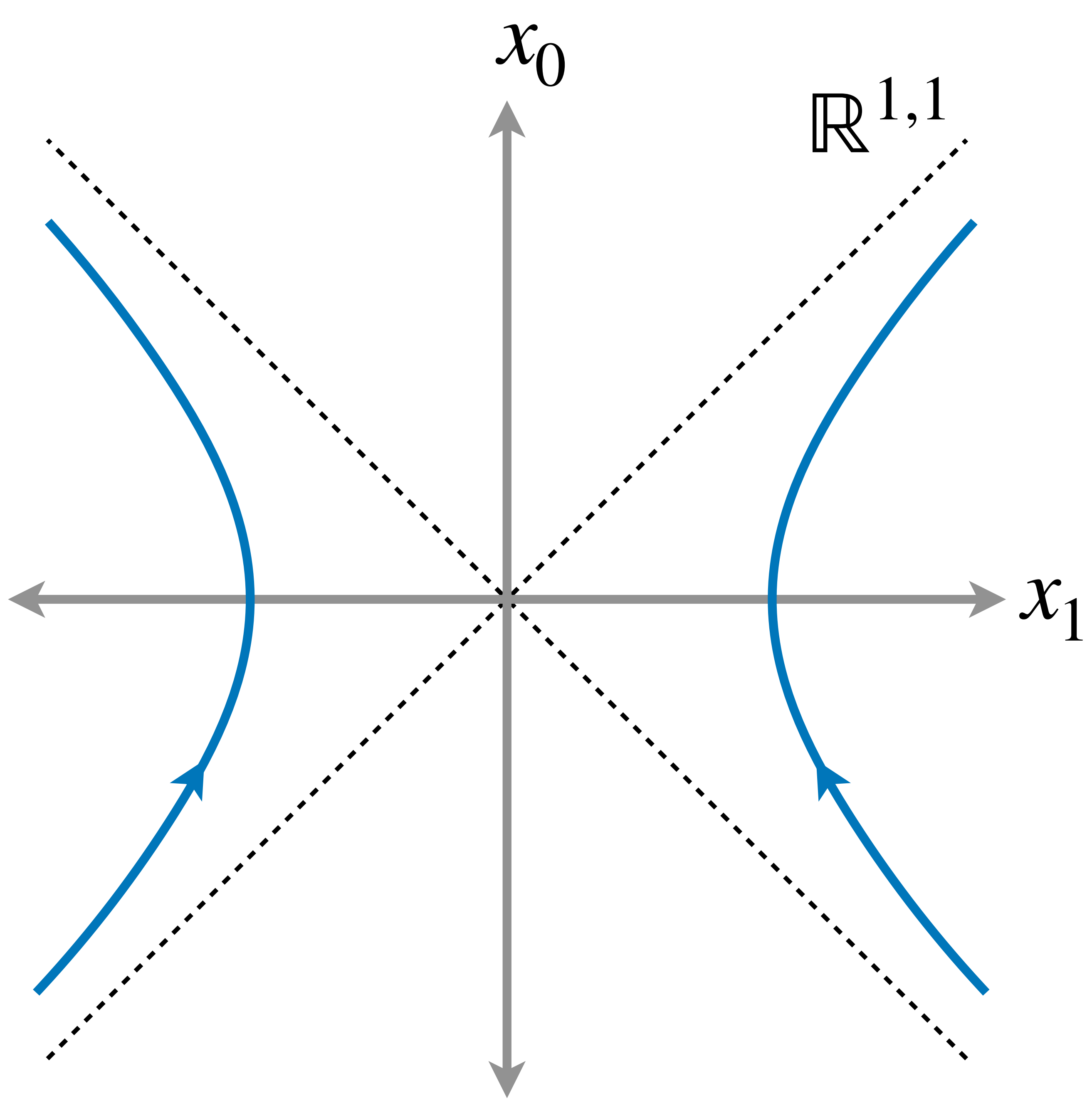
    [Banerjee, Kundu, Poojary '18; Vegh, 19; Gutiez, Hoyos '22]
- How do we compute string correlators in the conformal gauge?

Scattering on AdS_2 "wormhole"



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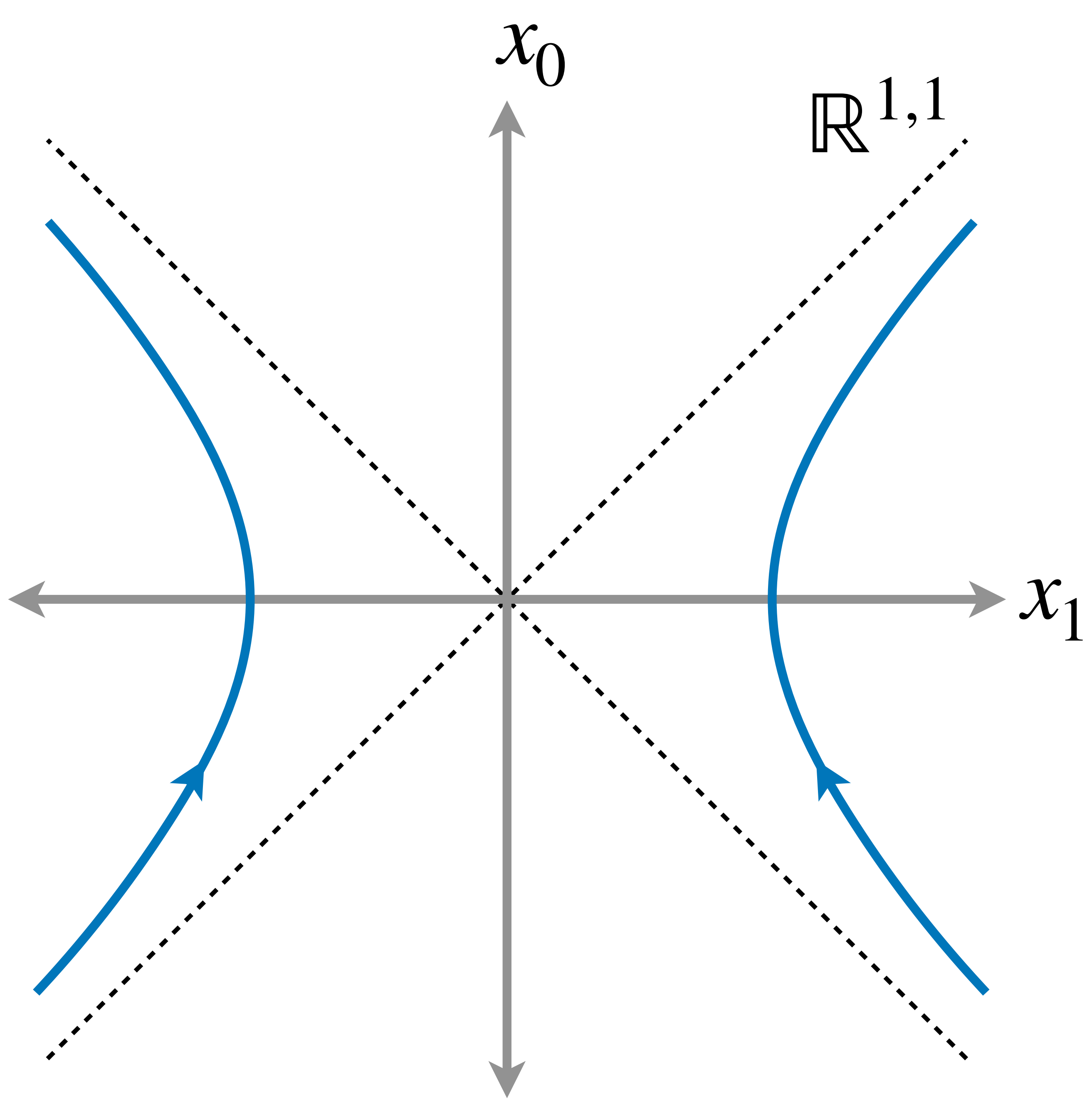

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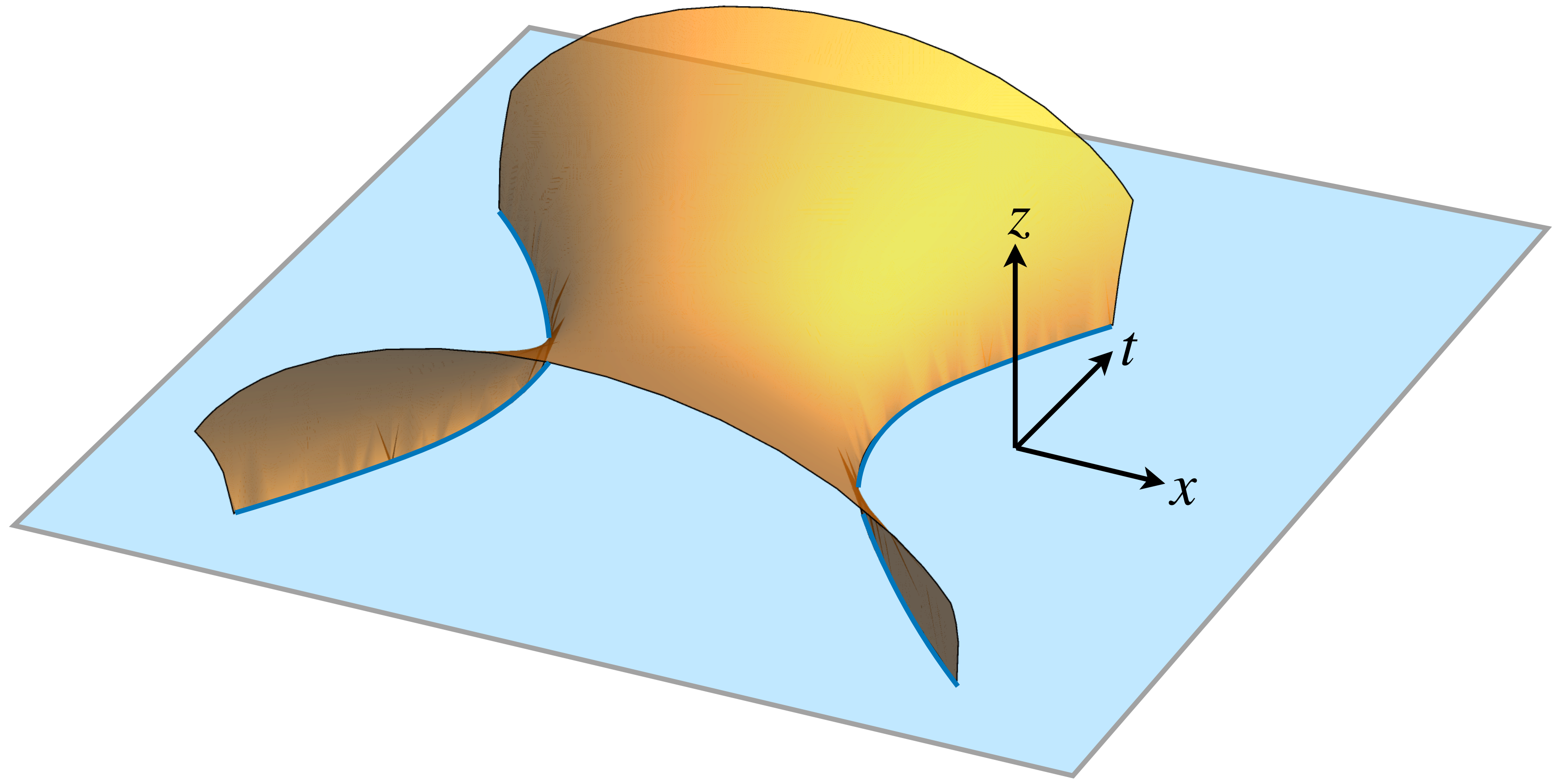
[Xiao '08; Jensen, Karch '13; Sonner '13, ...]



$$-x_0^2 + x_1^2 + z^2 = 1$$

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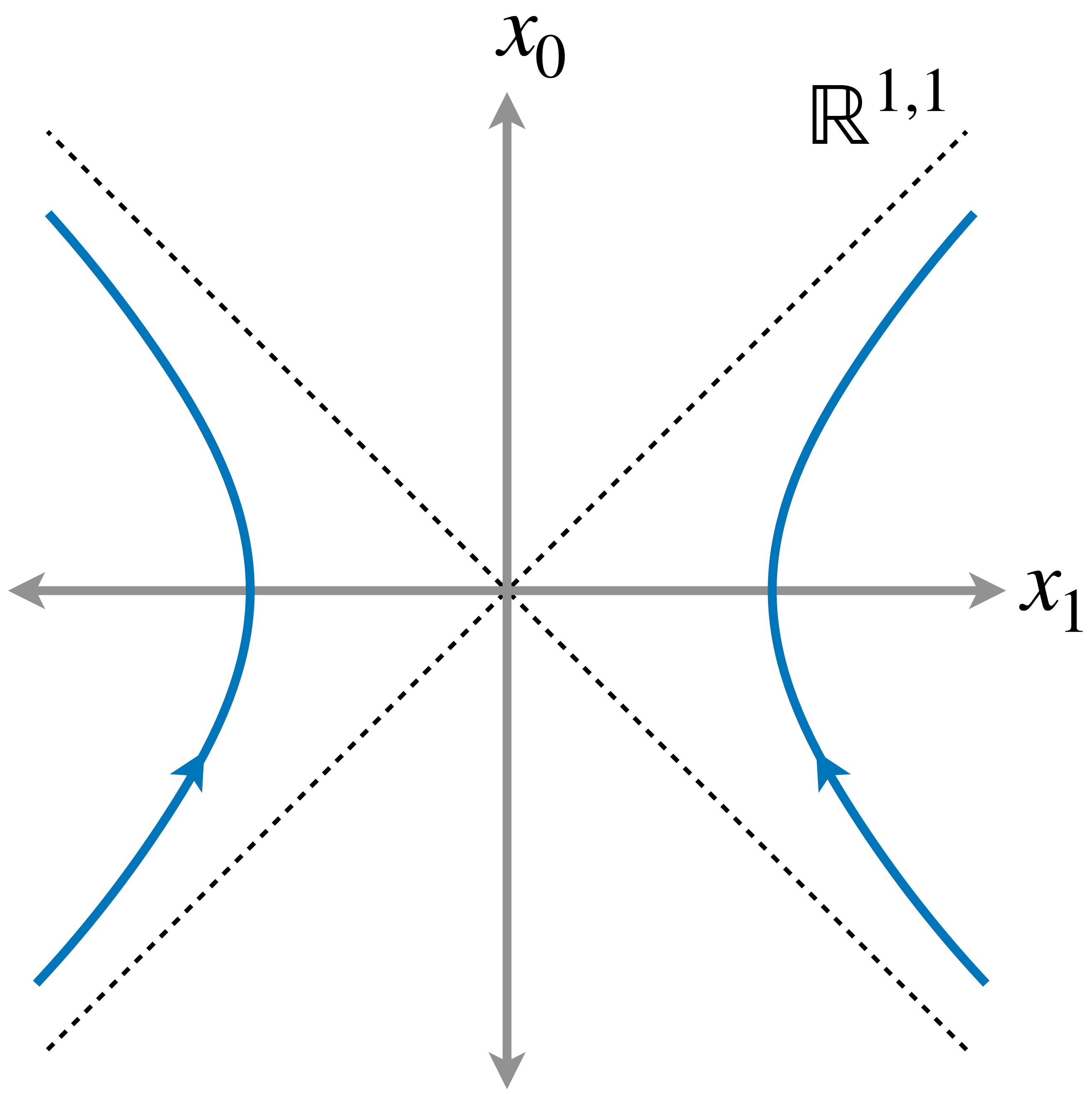

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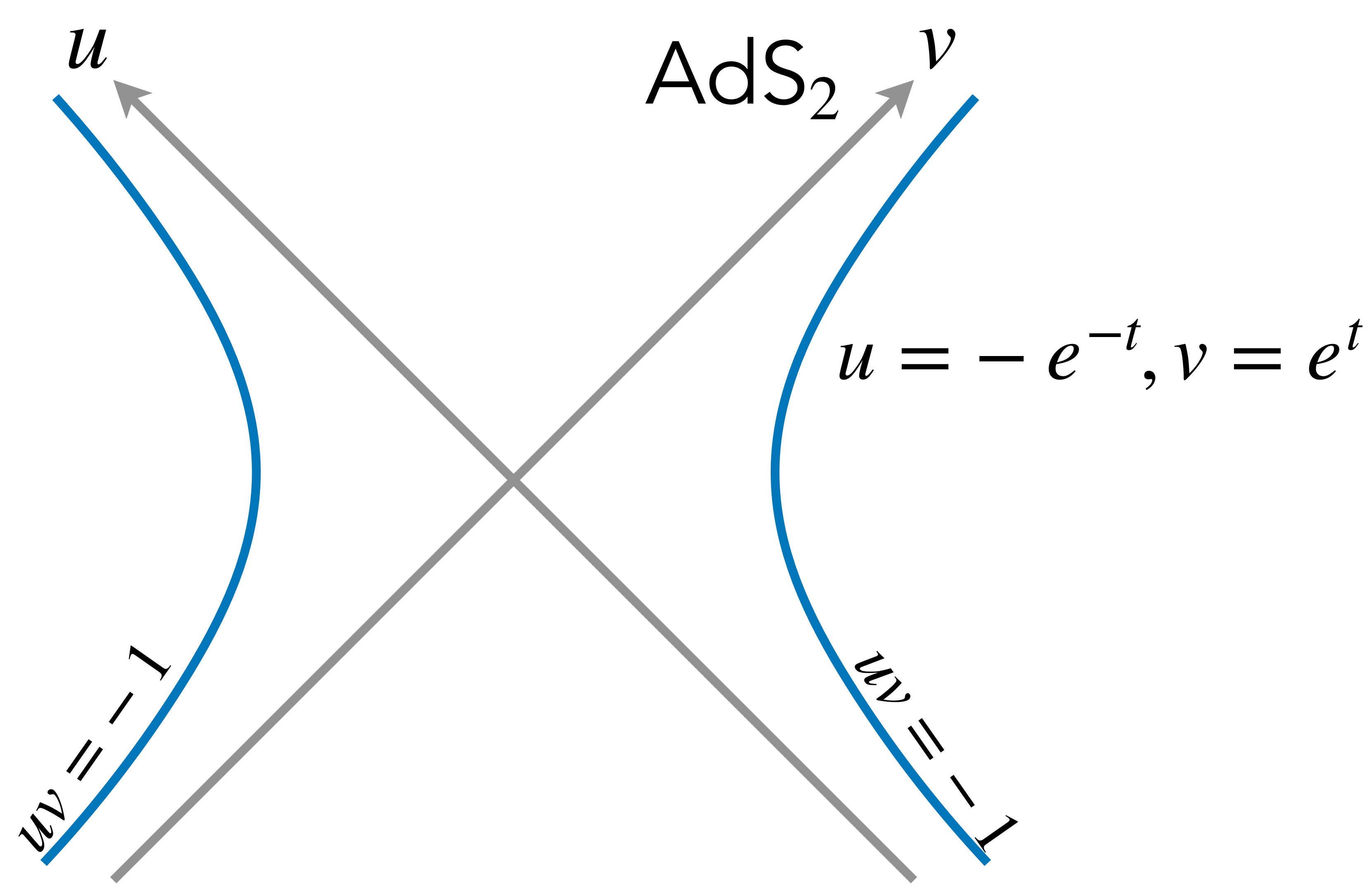

$$x_0 = \frac{u+v}{1-uv}, \quad x_1 = \frac{v-u}{1-uv}, \quad z = \frac{1+uv}{1-uv}$$

$$ds^2 = -\frac{4dudv}{(1+uv)^2}$$

OTOC as a scattering amplitude

We begin by writing the OTOC as the overlap of two states:

[Shenker, Stanford, '14;
de Boer, Llabres, Pedraza, Vegh '17]

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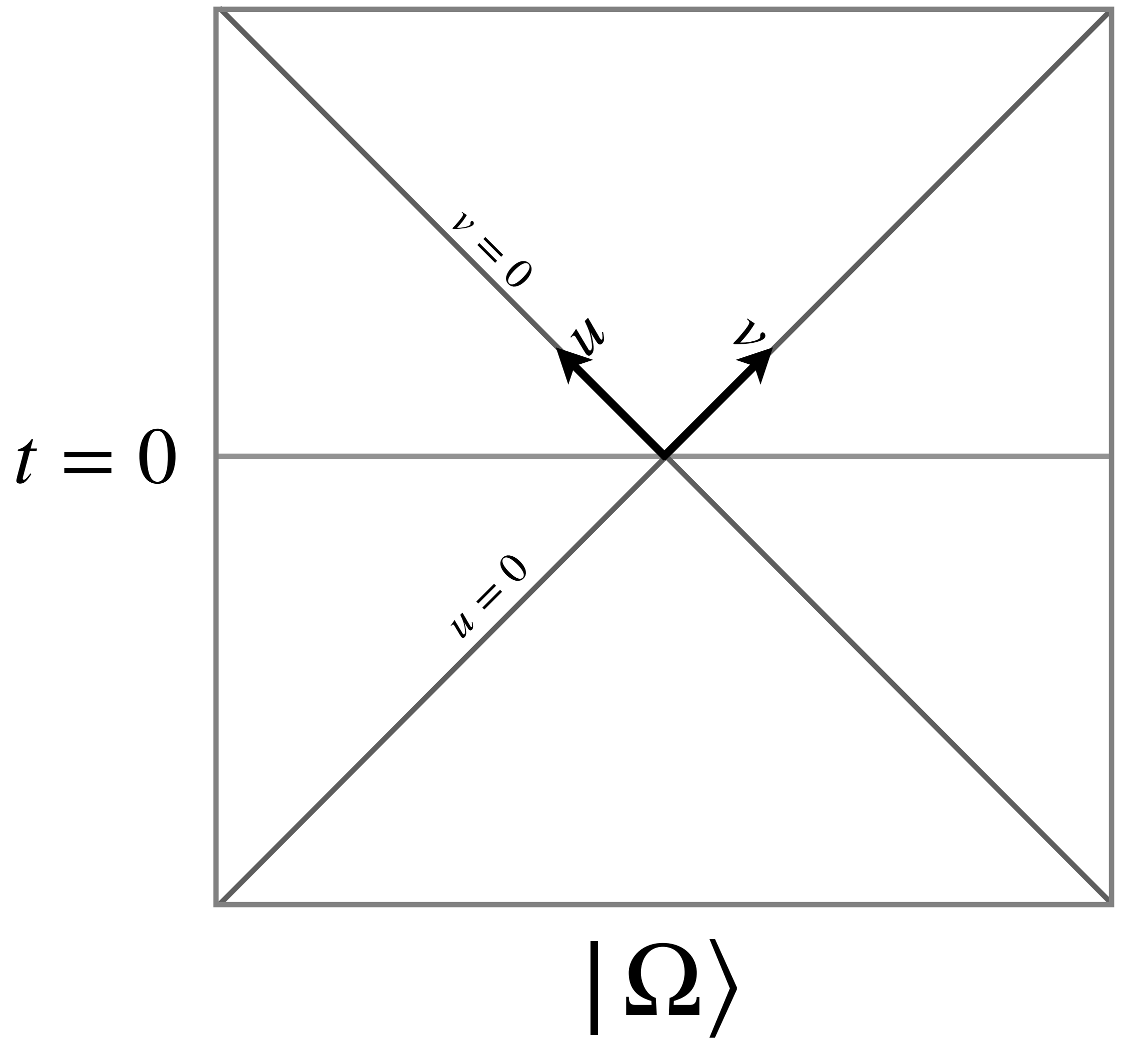
$$\Big\langle V(-\frac{t}{2})W(\frac{t}{2})V(-\frac{t}{2})W(\frac{t}{2})\Big\rangle = \langle\alpha|\beta\rangle,$$

Where $|\alpha\rangle = W(\frac{t}{2})V(-\frac{t}{2})\,|\Omega\rangle,$ and $|\beta\rangle = V(-\frac{t}{2})W(\frac{t}{2})\,|\Omega\rangle.$

These have the interpretation of "out" and "in" states for a scattering process on the AdS_2 worldsheet

OTOC as a scattering amplitude

$$|\alpha\rangle = W(t/2)V(-t/2)|\Omega\rangle$$



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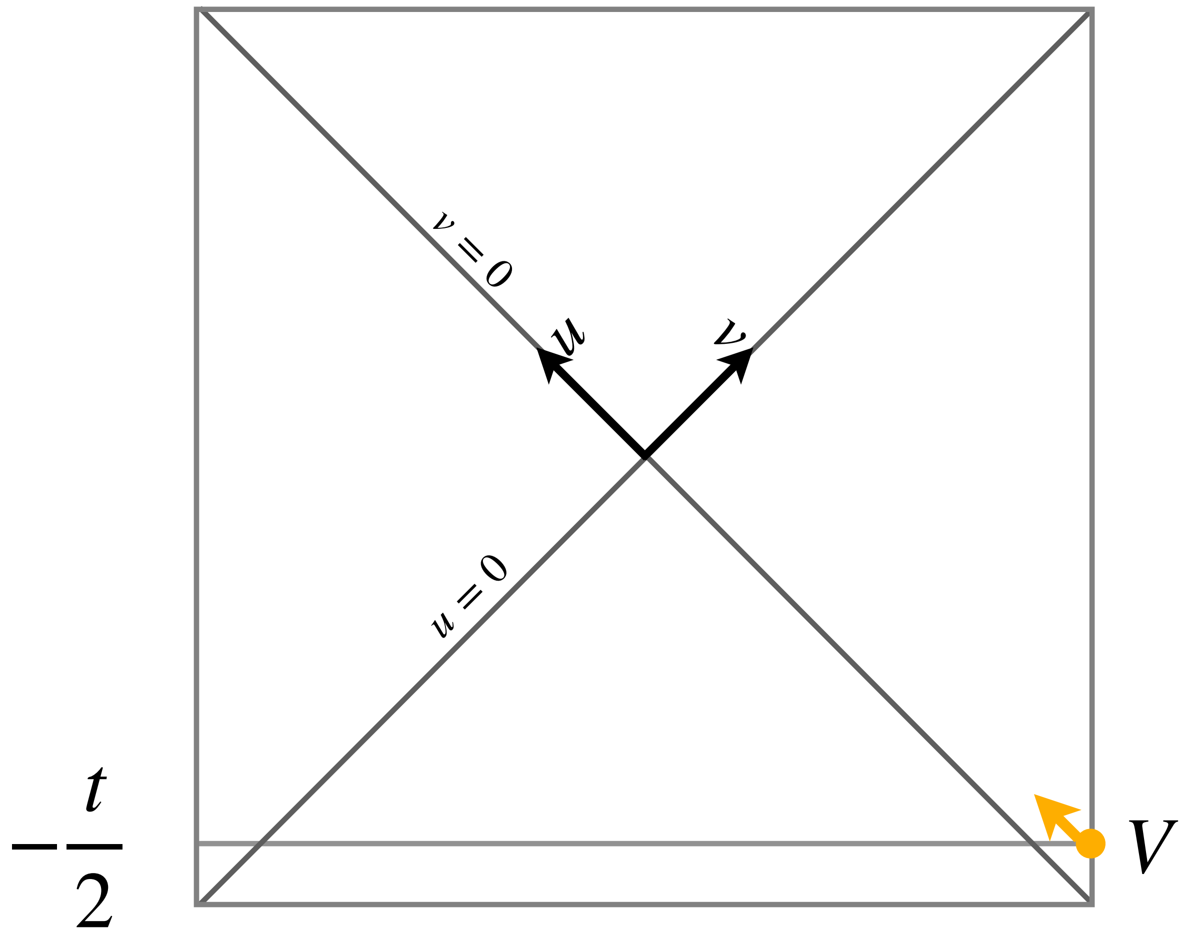

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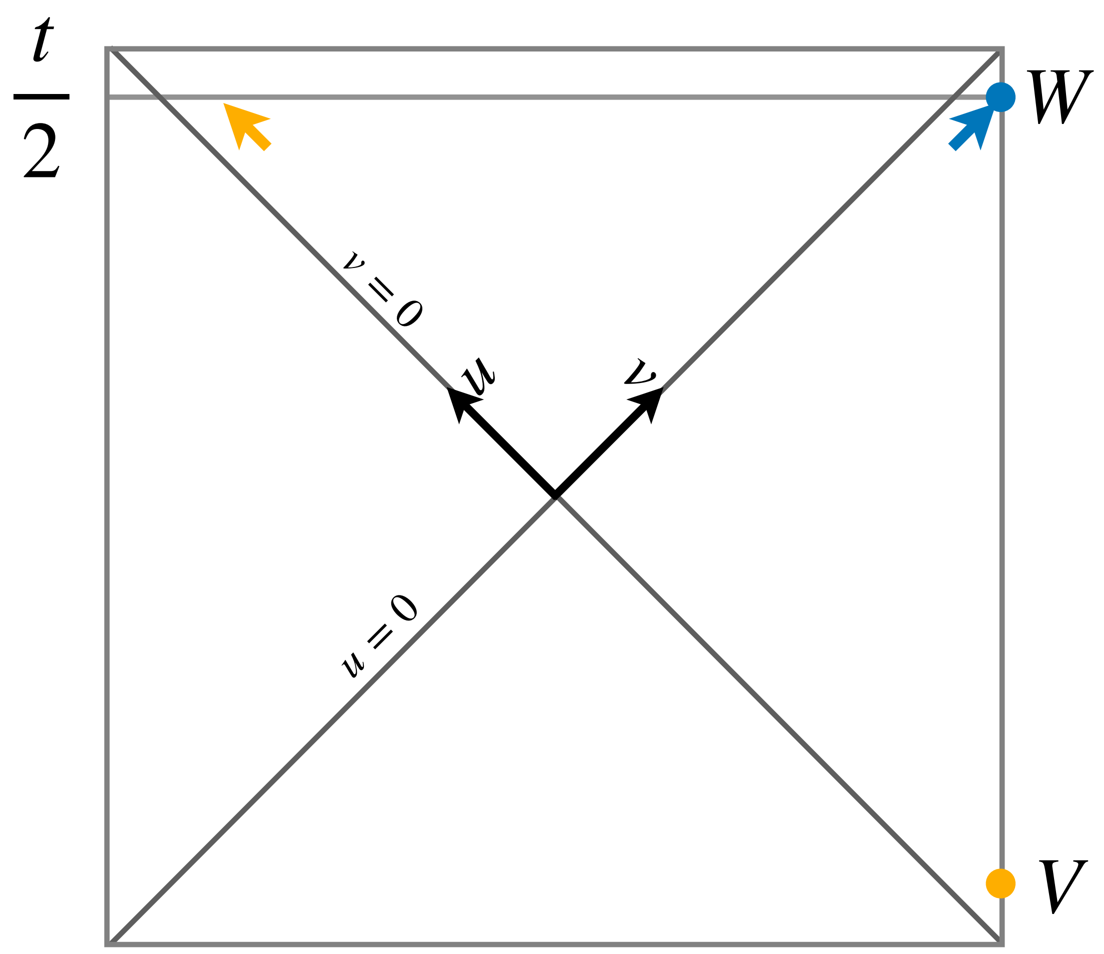

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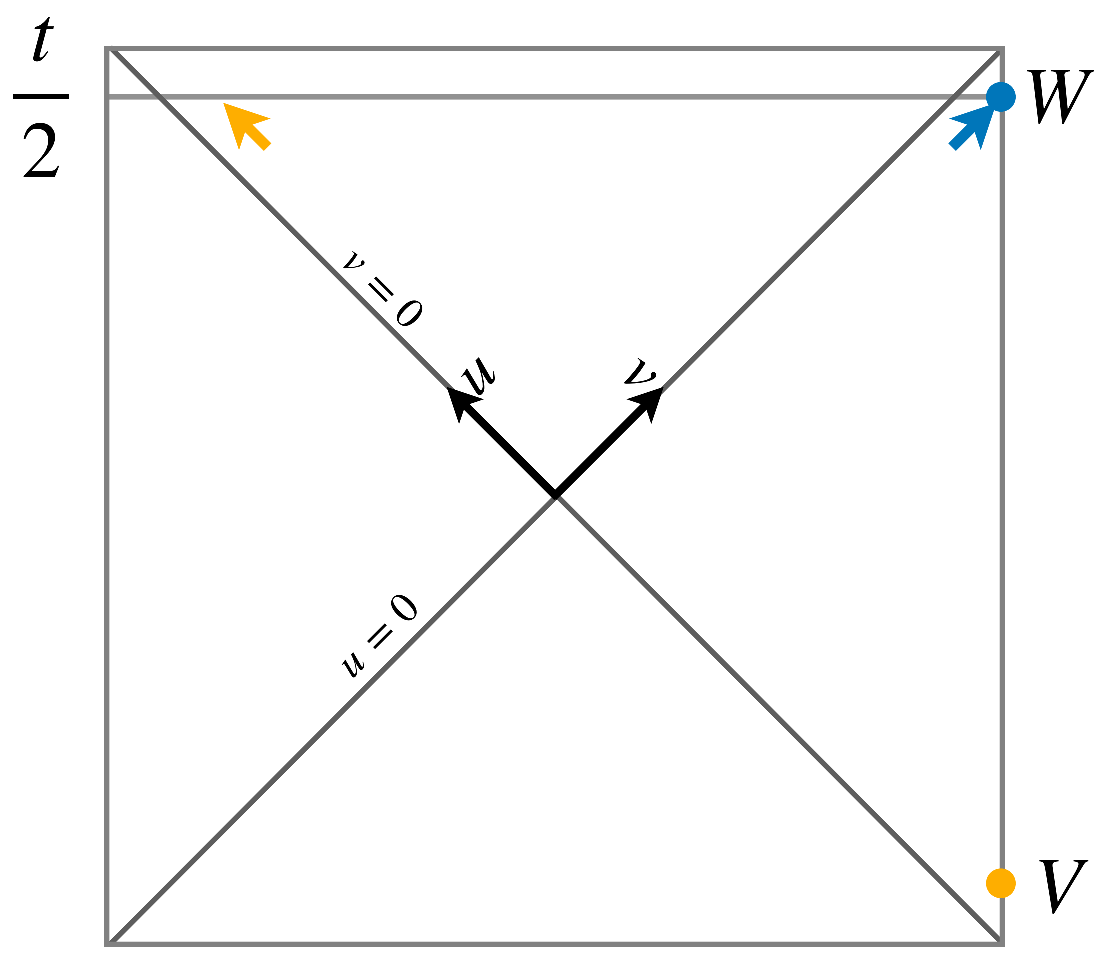

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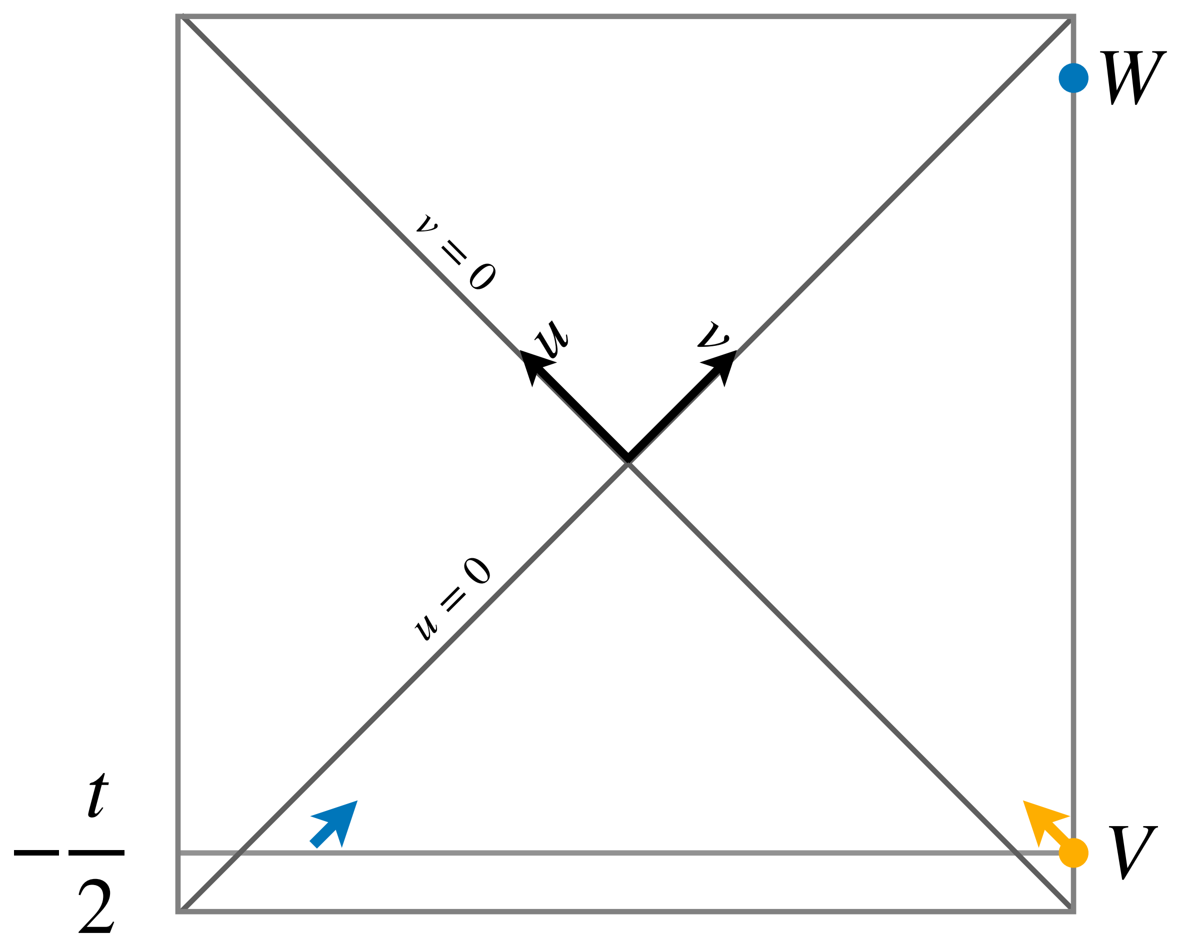

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We write the OTOC as a scattering amplitude : [Shenker, Stanford, '14; de Boer, Llabres, Pedraza, Vegh '17]

$$\langle V_1 W_3 V_2 W_4 \rangle = \int \prod_{i=1}^{4} dp_i \Psi_{\Delta_W}(p_1^u, t_1)^* \Phi_{\Delta_W}(p_3^v, t_3)^* \mathcal{S}(p_1^u, p_3^v; p_2^u, p_4^v) \Psi_{\Delta_V}(p_2^u, t_2) \Phi_{\Delta_W}(p_4^v, t_4),$$

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The wave functions are FTs of the bulk-boundary propagator, $K_\Delta = c_\Delta \left(\frac{(1+uv)}{(1+uv_i)(1+vu_i)} \right)^\Delta$:

$$\Psi_\Delta(p^u, t_i) = \int dv e^{2ip^u v} K_\Delta(0, v, t_i) = \theta(p^u) \frac{(2ip^u v_i)^\Delta}{\sqrt{\Gamma(2\Delta)} p^u},$$

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Here, $(u_i, v_i) = (-e^{-t_i}, e^{t_i})$.

OTOC as a scattering amplitude

We write the OTOC as a scattering amplitude : [Shenker, Stanford, '14; de Boer, Llabres, Pedraza, Vegh '17]

$$\langle V_1 W_3 V_2 W_4 \rangle = \int \prod_{i=1}^{4} dp_i \Psi_{\Delta_W}(p_1^u, t_1)^* \Phi_{\Delta_W}(p_3^v, t_3)^* \mathcal{S}(p_1^u, p_3^v; p_2^u, p_4^v) \Psi_{\Delta_V}(p_2^u, t_2) \Phi_{\Delta_W}(p_4^v, t_4),$$

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The S-matrix is diagonal: $\mathcal{S}(p_1^u, p_3^v; p_2^u, p_4^v) = p_1^u p_3^v \delta(p_1^u - p_2^u) \delta(p_3^v - p_4^v) e^{2i\delta(p_1^u, p_3^u)}$.

Scattering on the long flat string

[Dubovsky, Flauger, Gorbenko '12]

$$\boxed{e^{2i\delta(p_1^u, p_3^u)} = e^{i\ell_s^2 p^u p^v} = e^{\frac{i}{4} s \ell_s^2}}$$

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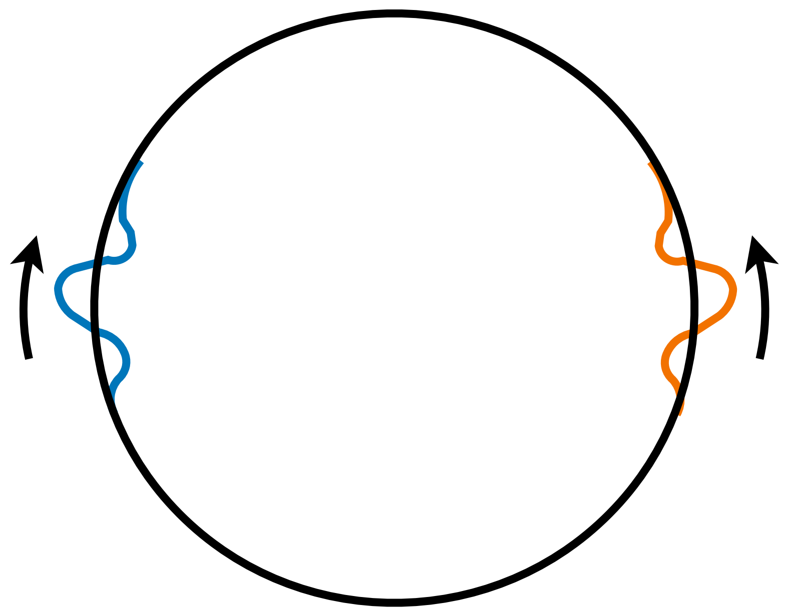
$$e^{-iE_{2p}t} = e^{-i2E_{1p}t + 2i\delta(s)\frac{2t}{R}}$$

$p^v = \frac{N_R}{R}$



$p^u = \frac{N_L}{R}$

$$E(N_L, N_R) = \sqrt{\frac{4\pi^2(N_L - N_R)^2}{R^2} + \frac{R^4}{\ell_s^4} + \frac{4\pi}{\ell_s^2}\left(N_L + N_R - \frac{D-2}{12}\right)}$$

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Essentially the same analysis holds in the case of OTOCs in JT gravity [Lam, Mertens, Turiaci, Verlinde '18]

Part III

Computing OTOCs in conformal gauge via the reparametrization mode

Lightning review of JT Gravity

Toy model of 2D gravity [Teiltelboim '83, Jackiw '85]. Dilaton ϕ, metric h, matter field y:

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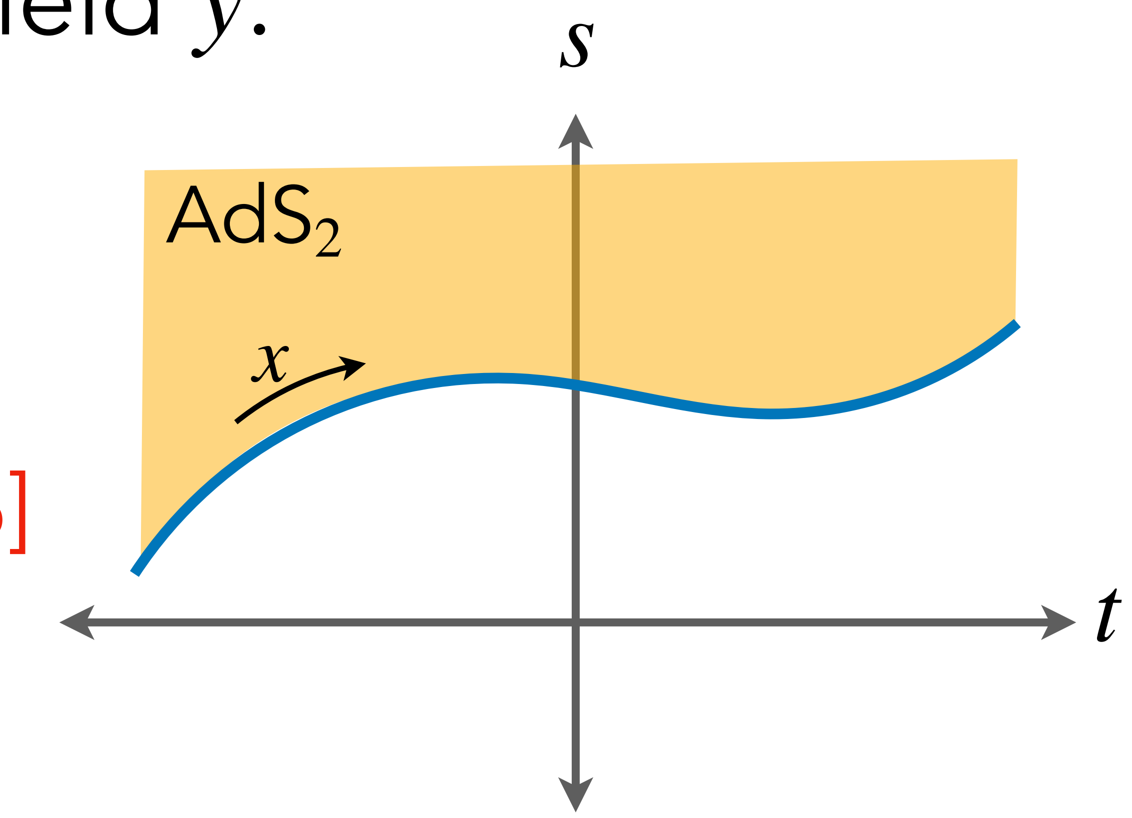

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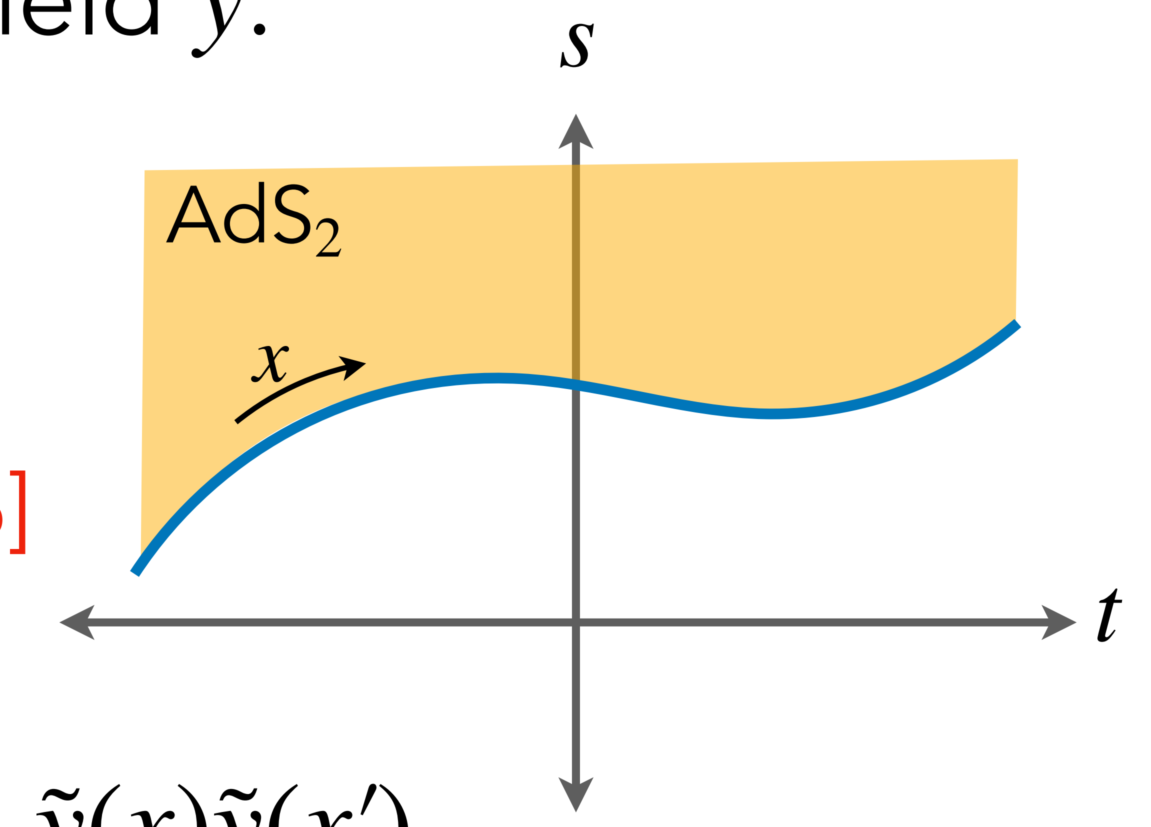

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$$S_{\mathrm{JT}}[\tilde{\phi}, t] = -\frac{\tilde{\phi}}{8\pi G_N}\int dx\{t, x\}, \qquad S_{\mathrm{m}}[\tilde{y}, t] = -\frac{D}{2}\int dx dx' \frac{\dot{t}(x)\dot{t}(x')}{(t(x) - t(x'))^2}\tilde{y}(x)\tilde{y}(x')$$

Where $\{t, x\} = \frac{\dddot{t}}{\dot{t}} - \frac{3}{2}\frac{\ddot{t}^2}{\dot{t}^2}$.

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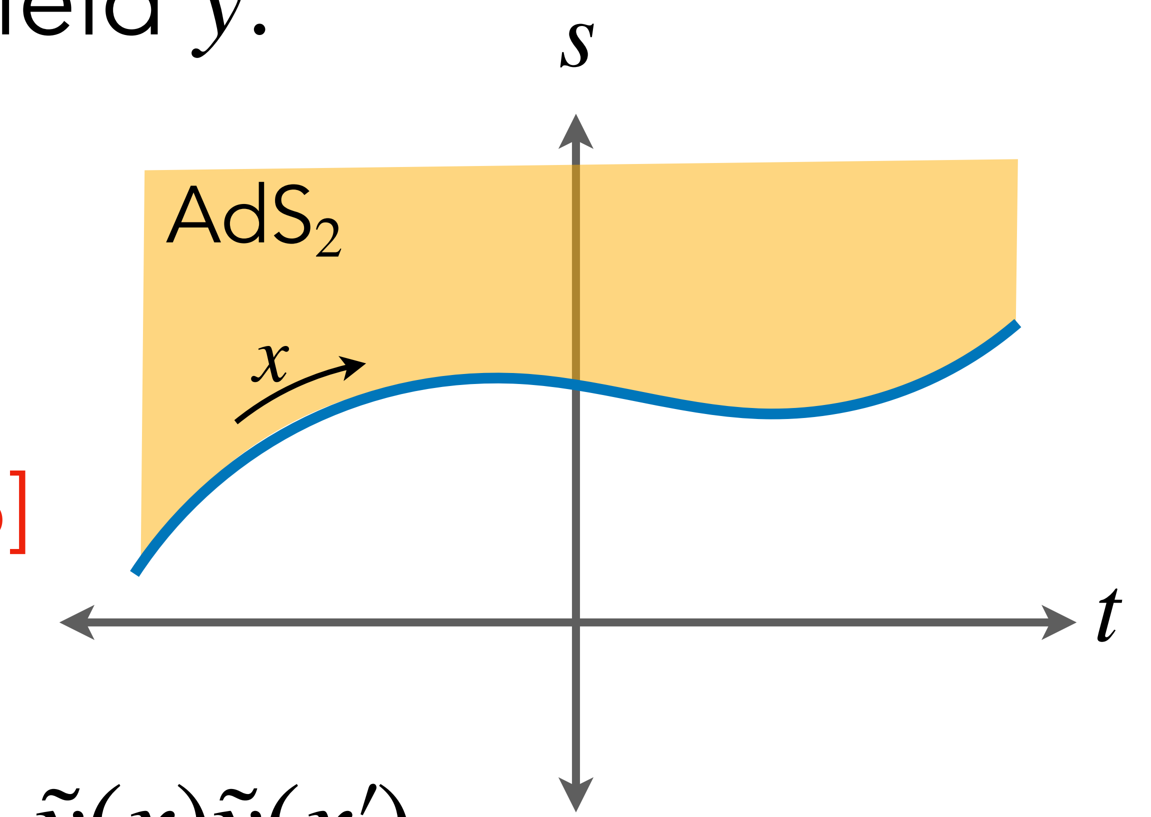

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Where $\{t, x\} = \frac{\dddot{t}}{\dot{t}} - \frac{3}{2}\frac{\ddot{t}^2}{\dot{t}^2}$. The partition function becomes:

$$Z[\tilde{y}] = \int \mathcal{D}h\mathcal{D}\phi\mathcal{D}y e^{-S[h,\phi,y]} = \int_{Diff(S^1)/SL(2,\mathbb{R})} \mathcal{D}t e^{-S_{\rm Schw.}[t(x)] - S_{\rm m}[\tilde{y}, t]}$$

[Kitaev; Almheiri, Polchinksi; Maldacena, Stanford, Yang; Jensen; Mertens, Engelsoy, Verlinde; Bagrets, Altland, Kamenev; Stanford, Witten; Mertens, Turiaci, Verlinde; Kitaev, Suh; …] [Mertens, Turiaci '22]

String action in conformal gauge

Integrals over boundary reparametrizations appear in conformal gauge

- In flat space [Polyakov '81, '86; Alvarez '83; Fradkin, Tseytlin '82; Cohen, Moore, Nelson, Polchinski '86, …]
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- Douglas integral for minimal area in flat space [Douglas '31]:

$$A = \underset{\alpha}{\text{minimize}} \left[\frac{1}{4\pi} \int_0^{2\pi} d\tau \int_0^{2\pi} d\tau' \frac{[\vec{x}(\alpha(\tau)) - \vec{x}(\alpha(\tau'))]^2}{\left[2\sin\left(\frac{\tau-\tau'}{2}\right)\right]^2} \right]$$

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We start with the Polyakov action:

$$S = \frac{T_s}{2} \int d^2\sigma \sqrt{h} h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu}(X)$$

Here $X^\mu = (x, z, y)$, $\quad G_{\mu\nu}(X) dX^\mu dX^\nu = \dfrac{dx^2 + dz^2}{z^2} + dy^2$, $\quad \sigma^\alpha = (s, t)$.

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Where $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$.

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The condition for the string $X^\mu(s, t)$ to be incident on the curve $\tilde{X}^\mu(\lambda)$ is:

$$X^\mu(0,t) = \tilde{X}^\mu(\alpha(t)),$$

where $\alpha(t)$ is a reparametrization of the boundary

String action in conformal gauge

Equations of motion:

$$0 = \frac{1}{T_s}\frac{\delta S}{\delta X^\mu} = \quad - \partial_\alpha(G_{\mu\nu}\partial^\alpha X^\nu) + \frac{1}{2}\partial_\mu G_{\nu\rho}\partial_\alpha X^\nu \partial^\alpha X^\rho$$

Subject to BC: $X^\mu(0,t) = \tilde{X}^\mu(\alpha(t))$

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Virasoro constraint = extremization over reparametrizations

Let $X^\mu(\sigma)$ solve the EOM with BC $X^\mu(0,t) = \tilde{X}^\mu(\alpha(t))$. If $\alpha(t) \to \alpha(t) + \delta\alpha(t)$, the variation in the on-shell action is:

$$\delta S = \int d\sigma^\alpha T_{\alpha\beta}\delta\sigma^\beta = \int dt T_{st}(0,t)\frac{\delta\alpha(t)}{\dot{\alpha}(t)}.$$

with $\delta s = 0$ and $\delta t = \dfrac{\delta\alpha(t)}{\dot{\alpha}(t)}$. This is somewhat analogous to Hamilton-Jacobi: $\delta S = p(t_f)\delta q(t_f) - p(t_i)\delta q(t_i)$.

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If f is holomorphic in the UHP with zero imaginary part on the real axis, then $f = 0$. Thus, we conclude:

$$T(s,t) = T_{\alpha\beta}(s,t) = 0.$$

The AdS$_2$ string in conformal gauge

Let's see how far we can get with the string in AdS$_2 \times S^1$:

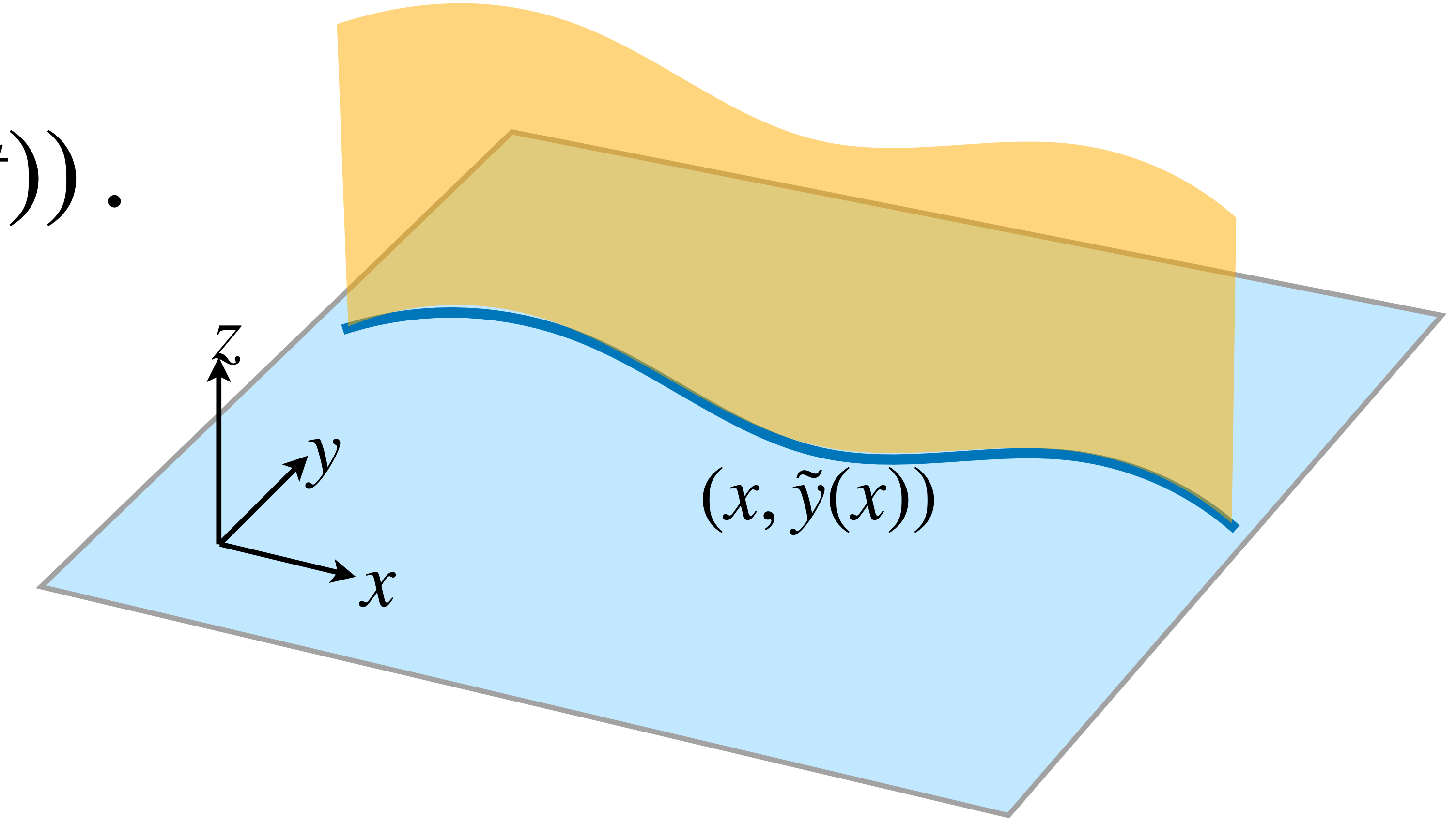
$$S = S_L[x, z] + S_T[y] + T_s A$$

Where

$$S_L[x,z] = \frac{T_s}{2}\int d^2\sigma \left[\frac{\partial_\alpha x \partial^\alpha x + \partial_\alpha z \partial^\alpha z}{z^2} - \frac{2}{s^2}\right], \qquad S_T[y] = \frac{T_s}{2}\int d^2\sigma \partial_\alpha y \partial^\alpha y, \qquad A = \int d^2\sigma \frac{1}{s^2}$$

And the BCs are:

$$x(0,t) = \alpha(t), \qquad z(0,t) = 0, \qquad y(0,t) = \tilde{y}(\alpha(t))\,.$$



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Where $K(s,t,t') = \frac{1}{\pi}\frac{s}{(t-t')^2}$. The action becomes:

$$S_T[\tilde{y}(\alpha(t))] = -\frac{T_s}{2\pi}\int dt dt' \frac{\tilde{y}(\alpha(t))\tilde{y}(\alpha(t))}{(t-t')^2} = \frac{T_s}{4\pi}\int dt dt' \frac{(\tilde{y}(\alpha(t)) - \tilde{y}(\alpha(t')))^2}{(t-t')^2}.$$

Longitudinal action

$$S_L[x,z] = \frac{T_s}{2}\int d^2\sigma \left[\frac{\partial_\alpha x \partial^\alpha x + \partial_\alpha z \partial^\alpha z}{z^2} - \frac{2}{s^2}\right], \qquad x(0,t) = \alpha(t), \quad z(0,t) = 0.$$

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EOM:

$$\partial_\alpha \left(\frac{1}{z^2} \partial^\alpha x \right) = 0, \qquad \partial_\alpha \left(\frac{1}{z^2} \partial^\alpha z \right) + \frac{1}{z^3} \left(\partial^\alpha x \partial_\alpha x + \partial^\alpha z \partial_\alpha z \right) = 0.$$

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Expanding $\alpha(t) = t + \epsilon(t)$, then to quadratic order in $\epsilon(t)$: [Polyakov Rychkov '00; Rychkov '02; Makeenko, Ambjorn '11]

$$S_L = \frac{6T_s}{\pi}\int dt dt' \frac{\epsilon(t)\epsilon(t')}{(t-t')^4} \qquad = \frac{T_s}{2\pi}\int dt dt' \frac{(\dot\epsilon(t) - \dot\epsilon(t'))^2}{(t-t')^2}$$

Setting $\epsilon(t) = \int \frac{d\omega}{2\pi} e^{-i\omega t}\epsilon(\omega)$, in Fourier space this becomes

$$S_L = \frac{T_s}{2\pi}\int d\omega \epsilon(\omega)\epsilon(-\omega)\,|\,\omega\,|^3.$$

Longitudinal action: general properties

$$S_L[x,z] = \frac{T_s}{2}\int d^2\sigma \left[\frac{\partial_\alpha x \partial^\alpha x + \partial_\alpha z \partial^\alpha z}{z^2} - \frac{2}{s^2}\right], \; x(0,t) = \alpha(t), \quad z(0,t) = 0$$

Symmetries: $SL(2,\mathbb{R})_L \times SL(2,\mathbb{R})_R$

$$SL(2,\mathbb{R})_L : x + iz \to f(x+iz), \qquad SL(2,\mathbb{R})_R : t + is \to f(t+is),$$

Where $f(w) = \dfrac{aw+b}{cw+d}$, $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$.

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Where $f(w) = \dfrac{aw+b}{cw+d}$, $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$.

Solution near the boundary [Polyakov, Rychkov '00]:

$$x(s,t) = \alpha(t) - \frac{\ddot{\alpha}(t)}{2}s^2 + \frac{g(t)}{3}s^3 + O(s^4), \qquad z(s,t) = \dot{\alpha}(t)s + \frac{h(t)}{3}s^3 + O(s^4).$$

Longitudinal action: perturbative analysis

With transverse modes turned off ($\tilde{y} = 0$), extremizing $\alpha(t)$ imposes:

$$\alpha(t) = \frac{at + b}{ct + d}, \qquad x + iz = \frac{a(t + is) + b}{c(t + is) + d}.$$

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This is because the Virasoro constraint without transverse modes becomes:

$$0 = T^L_{\alpha\beta} = \frac{\partial_\alpha x \partial_\beta x + \partial_\alpha z \partial_\beta z}{z^2} - \frac{1}{2}\delta_{\alpha\beta}\frac{\partial^\gamma x \partial_\gamma x + \partial^\gamma z \partial_\gamma z}{z^2}$$

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Letting $X = x + iz$, and $T^L = T^L_{tt} + iT^L_{st}$, this can be written as

$$0 = T^L = -\frac{8\partial X \partial \bar{X}}{(X - \bar{X})^2}.$$

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So $\bar{\partial} X = 0$ (or $\partial X = 0$).

Longitudinal action: perturbative analysis

Let $\alpha(t) = t + \epsilon(t), \quad x(s,t) = t + \xi(s,t), \quad z(s,t) = s + \zeta(s,t).$

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$$0 = s(\ddot{\xi} + \xi'') - 2(\xi' + \dot{\zeta}), \qquad 0 = s(\ddot{\zeta} + \zeta'') + 2(\dot{\xi} - \zeta'),$$

With BCs: $\xi(0,t) = \epsilon(t)$, $\zeta(0,t) = 0$.

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With BCs: $\xi(0,t) = \epsilon(t), \quad \zeta(0,t) = 0$. The solutions are:

$$\xi(s,t) = \int dt' K_x(s,t,t')\epsilon(t'), \qquad \zeta(s,t) = \int dt' K_z(s,t,t')\epsilon(t'),$$

Where

$$K_x(s,t,t') = \frac{4}{\pi}\frac{s^3(s^2 - (t-t')^2)}{(s^2 + (t-t')^2)^3}, \qquad K_z(s,t,t') = -\frac{8}{\pi}\frac{s^4(t-t')}{(s^2 + (t-t')^2)^3}.$$

Longitudinal action: perturbative analysis

The longitudinal action becomes:

$$S_L = T_s \int d^2\sigma \left[\frac{\dot{\xi}}{s^2} + \frac{z'}{s^2} - \frac{2\zeta}{s^3} - \frac{2\zeta\zeta'}{s^3} + \frac{3\zeta^2}{s^4} - \frac{2\zeta\dot{\xi}}{s^3} + \frac{\partial^\alpha \xi \partial_\alpha \xi + \partial^\alpha \zeta \partial_\alpha \zeta}{2s^2} + \dots \right]$$

↓

Integration by parts

$$x(s,t) = \alpha(t) - \frac{\ddot{\alpha}(t)}{2} s^2 + O(s^3), \quad z(s,t) = \dot{\alpha}(t) s + O(s^3).$$

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$$x(s,t) = \alpha(t) - \frac{\ddot{\alpha}(t)}{2} s^2 + O(s^3), \qquad z(s,t) = \dot{\alpha}(t) s + O(s^3).$$

$$= -\frac{T_s}{4} \int dt \xi(0,t) \xi'''(0,t) \qquad = -\frac{T_s}{4} \lim_{s \to 0^+} \int dt dt' \epsilon(t) \epsilon(t') K_x'''(s,t,t')$$

Longitudinal action: perturbative analysis

The longitudinal action becomes:

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$$= \frac{6T_s}{\pi} \int dt dt' \frac{\epsilon(t)\epsilon(t')}{(t-t')^4} \qquad = \frac{T_s}{2\pi} \int dt dt' \frac{(\dot{\epsilon}(t) - \dot{\epsilon}(t'))^2}{(t-t')^2}.$$

From line to circle

We can map from the line to the circle by setting $t = \tan(\tau/2)$, $\alpha(t) = \tan(\tilde{\alpha}(\tau)/2)$.

Expanding $\alpha(t) = t + \epsilon(t)$, $\tilde{\alpha}(\tau) = \tau + \tilde{\epsilon}(\tau)$, we have $\tilde{\epsilon}(\tau) = 2\cos^2\big(\tau/2\big)\epsilon(t) + O(\epsilon^2)$.

This gives the transverse action:

$$S_T[\tilde{y}(\tilde{\alpha}(\tau))] = -\frac{T_s}{2\pi}\int d\tau d\tau' \frac{\tilde{y}(\tilde{\alpha}(\tau))\tilde{y}(\tilde{\alpha}(\tau))}{\left[2\sin\left(\frac{\tau-\tau'}{2}\right)\right]^2} \qquad = \frac{T_s}{4\pi}\int d\tau d\tau' \frac{[\tilde{y}(\tilde{\alpha}(\tau)) - \tilde{y}(\tilde{\alpha}(\tau))]^2}{\left[2\sin\left(\frac{\tau-\tau'}{2}\right)\right]^2}.$$

And the longitudinal action:

$$S_L = \frac{6T_s}{\pi}\int d\tau d\tau' \frac{\tilde{\epsilon}(\tau)\tilde{\epsilon}(\tau')}{\left[2\sin\left(\frac{\tau-\tau'}{2}\right)\right]^4} \qquad = \frac{T_s}{2\pi}\int dt dt' \frac{(\dot{\epsilon}(\tau) - \dot{\epsilon}(\tau'))^2 - (\epsilon(\tau) - \epsilon(\tau'))^2}{\left[2\sin\left(\frac{\tau-\tau'}{2}\right)\right]^2}.$$

Letting $\epsilon(\tau) = \sum_{n\in\mathbb{Z}} \epsilon_n e^{-in\tau}$, we find the longitudinal action in Fourier space:

$$S_L = 4\pi T_s \sum_{n=2}^{\infty} |n|(n^2-1)\epsilon_n\epsilon_{-n}.$$

Note: $n = 0, \pm 1$ are zero modes.

Reparametrization path integral

The result of the classical analysis of the string in conformal gauge is:

$$Z[\tilde{y}] \approx e^{-S_{\mathrm{cl}}[\tilde{y}]} = \underset{\alpha}{\text{extremize}}\{e^{-S_L[\alpha]-S_T[\tilde{y}(\alpha)]}\}$$

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Now promote this to an integral over reparametrizations [Rychkov '02; Ambjorn, Makeenko '12, …]:

$$Z[\tilde{y}] \sim \int \mathscr{D}\alpha e^{-S_L[\alpha]-S_T[\tilde{y}(\alpha)]}$$

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This only captures a part of the full string non-linear sigma model partition function, which is schematically given by [Polyakov, …]:

$$Z[\tilde{y}] = \int \mathscr{D}h\mathscr{D}z\mathscr{D}x\mathscr{D}y\mathscr{D}\theta e^{-S} = \int \mathscr{D}\alpha \int \mathscr{D}b\mathscr{D}c\mathscr{D}x\mathscr{D}z\mathscr{D}y\mathscr{D}\theta e^{-S-S_{\rm gf}-S_{\rm ghost}}$$

Nonetheless, let's see what we can get out of the rep. path integral.

Correlators in the rep. path integral

Recall that $\langle y(\theta_1)\ldots\rangle = \frac{1}{Z}\left(\frac{\delta}{\delta\tilde{y}(\theta_1)}\ldots\right)Z[\tilde{y}]\Big|_{\tilde{y}=0}$. $\tilde{y}$ only appears in $S_T[\tilde{y}(\alpha)] = -\frac{T_s}{2\pi}\int d\tau d\tau' \frac{\tilde{y}(\tilde{\alpha}(\tau))\tilde{y}(\tilde{\alpha}(\tau))}{\left[2\sin\left(\frac{\tau-\tau'}{2}\right)\right]^2}$, which satisfies:

$$\frac{\delta^2 S_T}{\delta\tilde{y}(\theta_1)\delta\tilde{y}(\theta_2)} = \frac{T_s}{\pi}\frac{\dot{\beta}(\theta_1)\dot{\beta}(\theta_2)}{[2\sin\left(\frac{\beta(\theta_1)-\beta(\theta_2)}{2}\right)]^2} \equiv \frac{T_s}{\pi}B(\theta_1,\theta_2)$$

Where $\alpha(\beta(\theta)) = \theta$, $\beta(\alpha(\tau)) = \tau$.

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Where $\alpha(\beta(\theta)) = \theta$, $\beta(\alpha(\tau)) = \tau$. Thus, two and four-point functions take the form:

$$\langle y(\theta_1)y(\theta_2)\rangle = \frac{T_s}{\pi}\int \mathscr{D}\alpha e^{-S_L[\alpha]}B(\theta_1,\theta_2),$$

$$\langle y(\theta_1)\ldots y(\theta_4)\rangle = \frac{T_s^2}{\pi^2}\int \mathscr{D}\alpha e^{-S_L[\alpha]}\left[B(\theta_1,\theta_2)B(\theta_3,\theta_4) + 2\text{ perms}\right].$$

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For convenience, we introduce "fictitious" operators V and W, which obey:

$$\langle V(\theta_1)V(\theta_2)\rangle = \int \mathscr{D}\alpha e^{-S_L[\alpha]}B(\theta_1,\theta_2)^{\Delta_V}, \quad \langle V(\theta_1)V(\theta_2)W(\theta_3)W(\theta_4)\rangle = \int \mathscr{D}\alpha e^{-S_L[\alpha]}B(\theta_1,\theta_2)^{\Delta_V}B(\theta_3,\theta_4)^{\Delta_W}.$$

$SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ in rep. path integral

Recall that we have two $SL(2,\mathbb{R})$ symmetries:

(1) $\alpha(t) \to \alpha\left(\dfrac{at+b}{ct+d}\right)$, $\beta(t) \to \dfrac{a'\beta(t)+b'}{c'\beta(t)+d'}$.

This is a gauge symmetry, and needs to be gauge fixed in the path integral. For our perturbative analysis, it is sufficient to write:

$$\mathscr{D}\alpha = d\epsilon_0 d\epsilon_1 d\epsilon_{-1} \prod_{n\geq 2} d\epsilon_n d\epsilon_{-n}$$

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This is a gauge symmetry, and needs to be gauge fixed in the path integral. For our perturbative analysis, it is sufficient to write:

$$\mathscr{D}\alpha = d\epsilon_0 d\epsilon_1 d\epsilon_{-1} \prod_{n\geq 2} d\epsilon_n d\epsilon_{-n}$$

(2) $\alpha(t) \to \frac{a\alpha(t)+b}{c\alpha(t)+d}$, $\beta(t) \to \beta\left(\frac{a't+b'}{ct'+d'}\right)$.

This is a physical symmetry and leads to Ward identities of the form:

$$\langle V(\theta_1)V(\theta_2)\rangle = \dot{f}(\theta_1)^{\Delta_V}\dot{f}(\theta_2)^{\Delta_V}\langle V(f(\theta_1))V(f(\theta_2))\rangle.$$

Perturbation theory in the reparametrization path integral

Let $\alpha(\tau) = \tau + \epsilon(\tau)$. Then $\beta(\theta) = \theta - \epsilon(\theta) + O(\epsilon^2)$, and

$$B(\theta_1, \theta_2) = \frac{\dot\beta(\theta_1)\dot\beta(\theta_2)}{[2\sin\left(\frac{\beta(\theta_1)-\beta(\theta_2)}{2}\right)]^2} = \frac{1}{[2\sin\frac{\theta_{12}}{2}]^2}\left(1 + \underbrace{\dot\epsilon(\theta_1) + \dot\epsilon(\theta_2) - (\epsilon(\theta_1) - \epsilon(\theta_2))\cot\frac{\theta_{12}}{2}}_{\equiv \mathscr{B}_1(\theta_1,\theta_2)} + O(\epsilon^2)\right).$$

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The ϵ propagator in Fourier space is given by (for $n \neq 0, \pm 1$):

$$\langle \epsilon_n \epsilon_m \rangle = \frac{1}{4\pi T_s |n|(n^2-1)}\delta_{n+m,0}.$$

In position space:

$$\langle \epsilon(\theta)\epsilon(0)\rangle = \frac{1}{4\pi T_s}\sum_{n\neq 0,\pm 1}\frac{e^{in\theta}}{|n|(n^2-1)} = \frac{1}{T_s}\left[a + b\cos\theta + \frac{1}{2\pi}\sin^2\frac{\theta}{2}\log\left(4\sin^2\frac{\theta}{2}\right)\right].$$

Perturbation theory in the reparametrization path integral

Let $\alpha(t) = t + \epsilon(t)$. Then $\beta(x) = x - \epsilon(x) + O(\epsilon^2)$, and

$$B(x_1, x_2) = \frac{\dot\beta(x_1)\dot\beta(x_2)}{(\beta(x_1) - \beta(x_2))^2} = \frac{1}{x_{12}^2}\Big(1 + \underbrace{\dot\epsilon(x_1) + \dot\epsilon(x_2) - \frac{2}{x_{12}}(\epsilon(x_1) - \epsilon(x_2))}_{\mathscr{B}_1(x_1,x_2)} + O(\epsilon^2)\Big).$$

The ϵ propagator in Fourier space is given by:

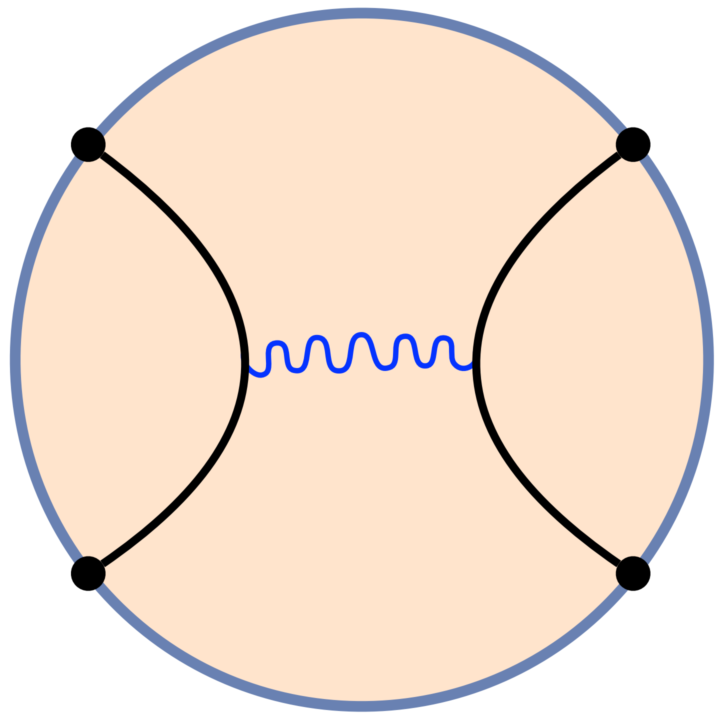
$$\langle \epsilon(\omega)\epsilon(\omega')\rangle = \frac{\pi}{T_s}\delta(\omega + \omega')\frac{1}{|\omega|^3}.$$

In position space:

$$\langle \epsilon(x)\epsilon(0)\rangle = \frac{1}{4\pi T_s}\int d\omega \frac{e^{-i\omega x}}{|\omega|^3} = \frac{1}{T_s}\left[a + bx^2 + \frac{1}{8\pi}x^2\log(x^2)\right].$$

Tree-level four-point function

$$\langle V(\theta_1)V(\theta_2)W(\theta_3)W(\theta_4)\rangle = \frac{1}{\left[2\sin\frac{\theta_{12}}{2}\right]^2\left[2\sin\frac{\theta_{34}}{2}\right]^2}\left(1 + \Delta_V\Delta_W\langle\mathscr{B}_1(\theta_1,\theta_2)\mathscr{B}_1(\theta_3,\theta_4)\rangle + O(1/T_s^2)\right),$$



Where

$$\langle\mathscr{B}_1(\theta_1,\theta_2)\mathscr{B}_1(\theta_3,\theta_4)\rangle = -\frac{1}{4\pi T_s}\left[4 + \frac{2-\chi}{\chi}\log\left((1-\chi)^2\right)\right].$$

Tree-level four-point function

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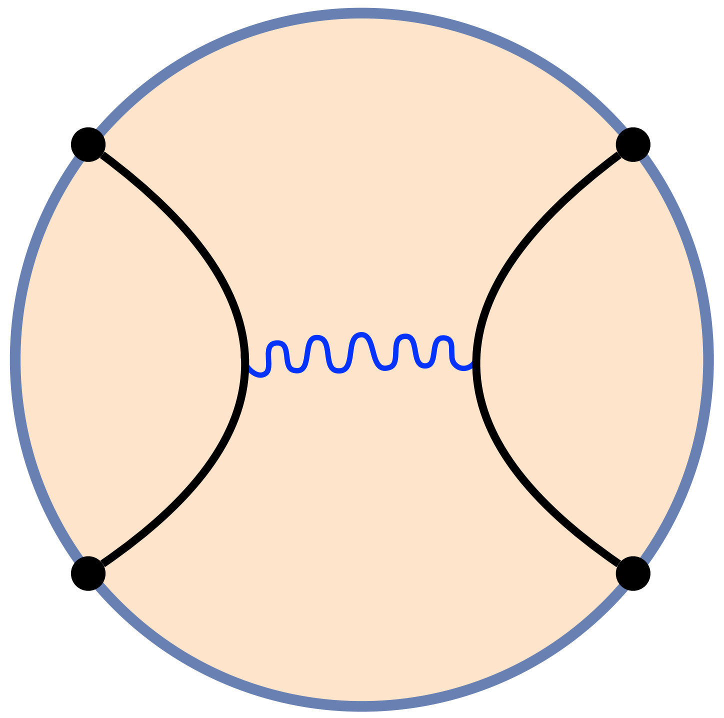


$$\langle\mathscr{B}_1(\theta_1,\theta_2)\mathscr{B}_1(\theta_3,\theta_4)\rangle = -\frac{1}{4\pi T_s}\left[4+\frac{2-\chi}{\chi}\log\left((1-\chi)^2\right)\right].$$

Setting $\Delta_V = \Delta_W = 1$ and writing:

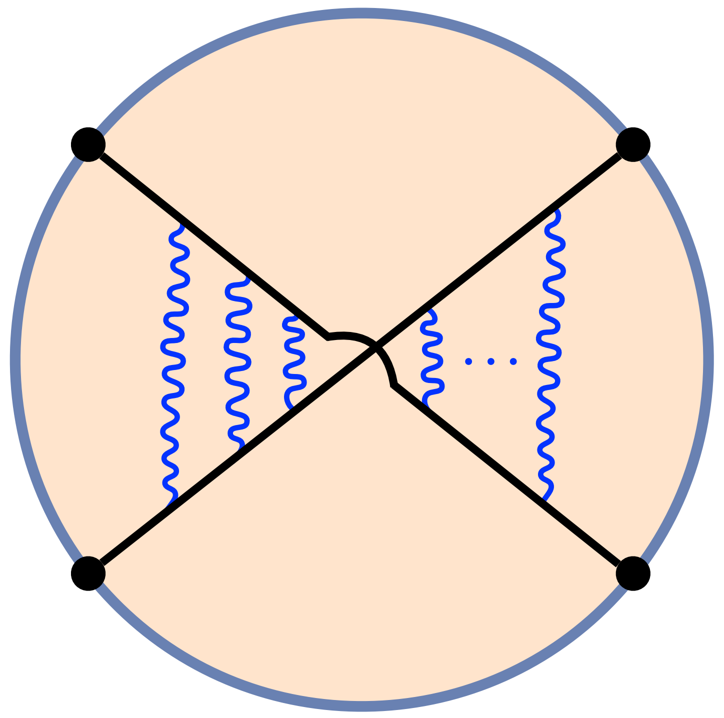
$$\langle y(\theta_1)y(\theta_2)y(\theta_3)y(\theta_4)\rangle = \langle V(\theta_1)V(\theta_2)W(\theta_3)W(\theta_4)\rangle + \text{ 2 permutations},$$

This reproduces the tree-level correlator in static gauge.

Doubled-scaled OTOC

Now we use the reparametrization path integral to compute the OTOC in the double scaling limit. We keep only the quadratic action. This corresponds to resuming all reparametrization mode exchanges.

The computation is easiest on the line, in which case we can write:



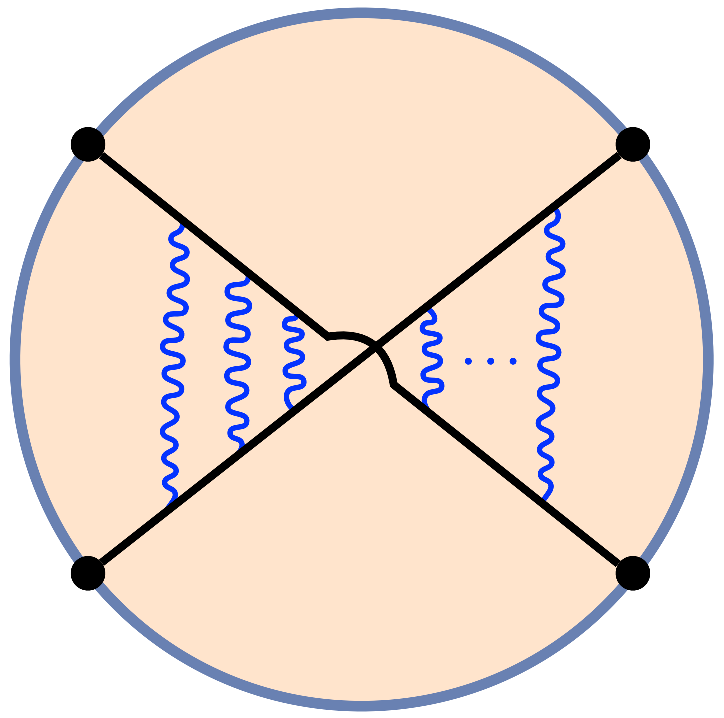
$$(1+\dot{\epsilon}_i)^{\Delta} = \frac{\partial^{\Delta}}{\partial \alpha_i^{\Delta}} e^{\alpha_i (1+\dot{\epsilon}_i)}, \qquad \frac{1}{(1+\frac{\epsilon_{ij}}{x_{ij}})^{2\Delta}} = \frac{1}{\Gamma(2\Delta)} \int_0^{\infty} dp p^{2\Delta-1} e^{-p(1+\frac{\epsilon_{ij}}{x_{ij}})}$$

Doubled-scaled OTOC

$$\frac{\langle V_1 V_2 W_3 W_4 \rangle}{\langle V_1 V_2 \rangle \langle W_3 W_4 \rangle} = \prod_i \frac{\partial^{\Delta_i}}{\partial \alpha_i^{\Delta_i}} \left[\prod_i e^{\alpha_i} \int_0^\infty \frac{p^{2\Delta_V - 1} dp}{\Gamma(2\Delta_V)} e^{-p} \int_0^\infty \frac{q^{2\Delta_V - 1} dq}{\Gamma(2\Delta_W)} e^{-q} \times X \right]$$

Where

$$X = \left\langle \exp\left(-p \frac{\epsilon_{12}}{x_{12}} - q \frac{\epsilon_{34}}{x_{34}} + \sum_{i=1}^{4} \alpha_i \dot{\epsilon}_i \right) \right\rangle = \exp \left\langle \frac{1}{2} \left(-p \frac{\epsilon_{12}}{x_{12}} - q \frac{\epsilon_{34}}{x_{34}} + \sum_{i=1}^{4} \alpha_i \dot{\epsilon}_i \right)^2 \right\rangle .$$

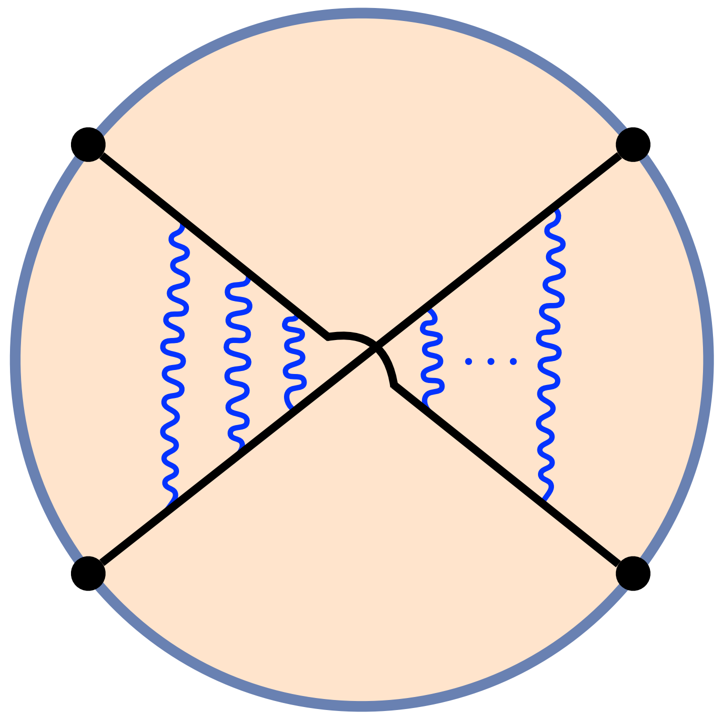


Doubled-scaled OTOC

$$\frac{\langle V_1V_2W_3W_4\rangle}{\langle V_1V_2\rangle\langle W_3W_4\rangle}=\prod_i\frac{\partial^{\Delta_i}}{\partial\alpha_i^{\Delta_i}}\left[\prod_i e^{\alpha_i}\int_0^\infty\frac{p^{2\Delta_V-1}dp}{\Gamma(2\Delta_V)}e^{-p}\int_0^\infty\frac{q^{2\Delta_V-1}dq}{\Gamma(2\Delta_W)}e^{-q}\times X\right]$$

Where

$$X=\left\langle \exp\left(-p\frac{\epsilon_{12}}{x_{12}}-q\frac{\epsilon_{34}}{x_{34}}+\sum_{i=1}^{4}\alpha_i\dot{\epsilon}_i\right)\right\rangle=\exp\left\langle\frac{1}{2}\left(-p\frac{\epsilon_{12}}{x_{12}}-q\frac{\epsilon_{34}}{x_{34}}+\sum_{i=1}^{4}\alpha_i\dot{\epsilon}_i\right)^2\right\rangle.$$



In the double scaling limit, $\log X=\frac{pq}{x_{12}x_{34}}\langle\epsilon_{12}\epsilon_{34}\rangle+\ldots\to-\kappa pq$ where $\frac{e^t}{16T_s}$. Therefore:

$$\frac{\langle V_1V_2W_3W_4\rangle}{\langle V_1V_2\rangle\langle W_3W_4\rangle}=\int_0^\infty\frac{p^{2\Delta_V-1}dp}{\Gamma(2\Delta_V)}e^{-p}\int_0^\infty\frac{q^{2\Delta_W-1}dq}{\Gamma(2\Delta_W)}e^{-q}e^{-\kappa pq}$$

$$=\kappa^{-2\Delta_V}U(2\Delta_V,1+2\Delta_V-2\Delta_W,\kappa^{-1}).$$

In agreement with the scattering result.

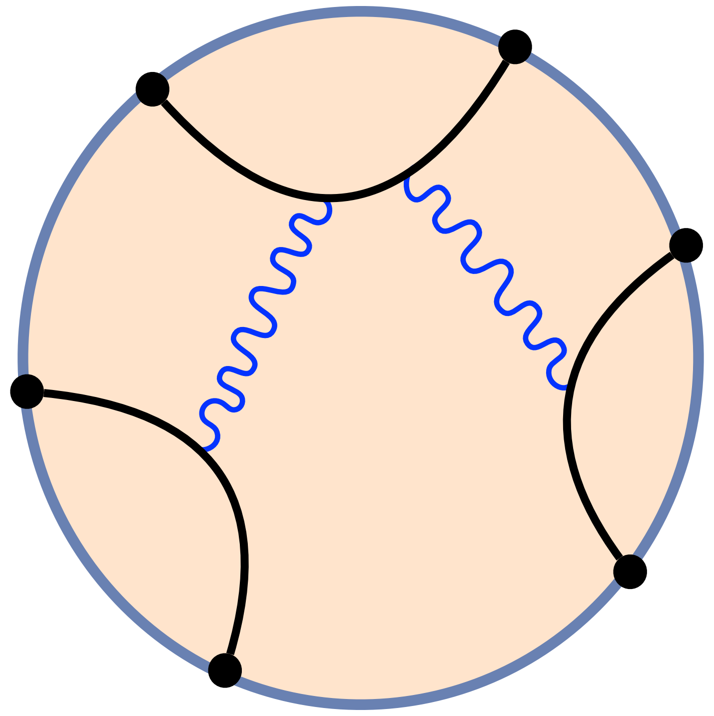
Summary

- We saw that the double scaled OTOC could be understood as a holographic scattering process on the string worldsheet in the manner of [SS' 14], with the interaction governed by the flat string S-matrix [DFG '12]. See also [de Boer, Llabres, Pedraza, Vegh '17].
- The scattering result agrees to 3-loops with the analytic result of [Ferrero, Meneghelli '21].
- We also clarified the role of the boundary reparametrization mode in the computation of the string boundary correlators in conformal gauge, and used it to derive the OTOC in the Lyapunov regime and (more heuristically) in the double scaling limit.

Open questions

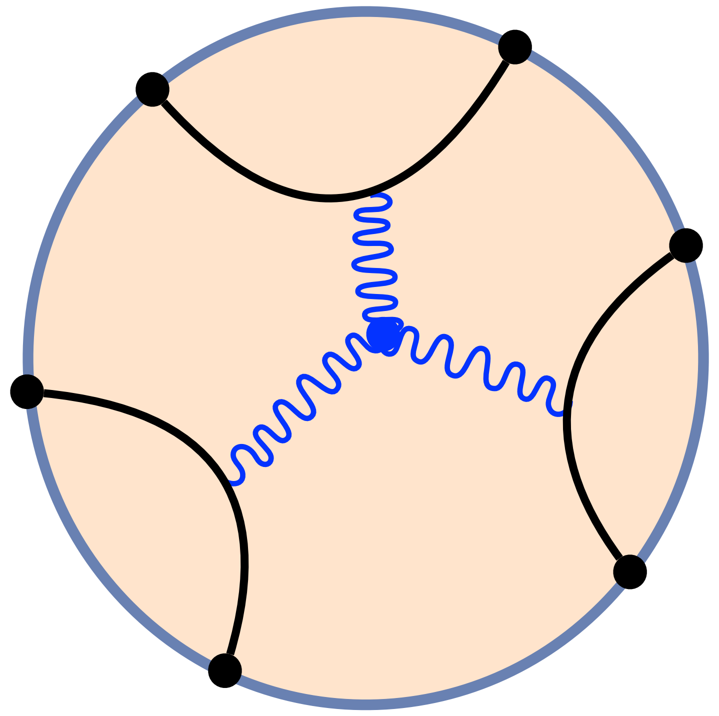
Can the reparametrization path integral be defined more precisely and used to compute other observables?

- Tree-level (classical) contributions to higher-point correlators [WIP w/ Giombi, Komatsu, Shan]
- Loop corrections to two- or four-point functions? Partition function?



Is there a similar reparametrization action for the AdS_2 string in AdS_3?

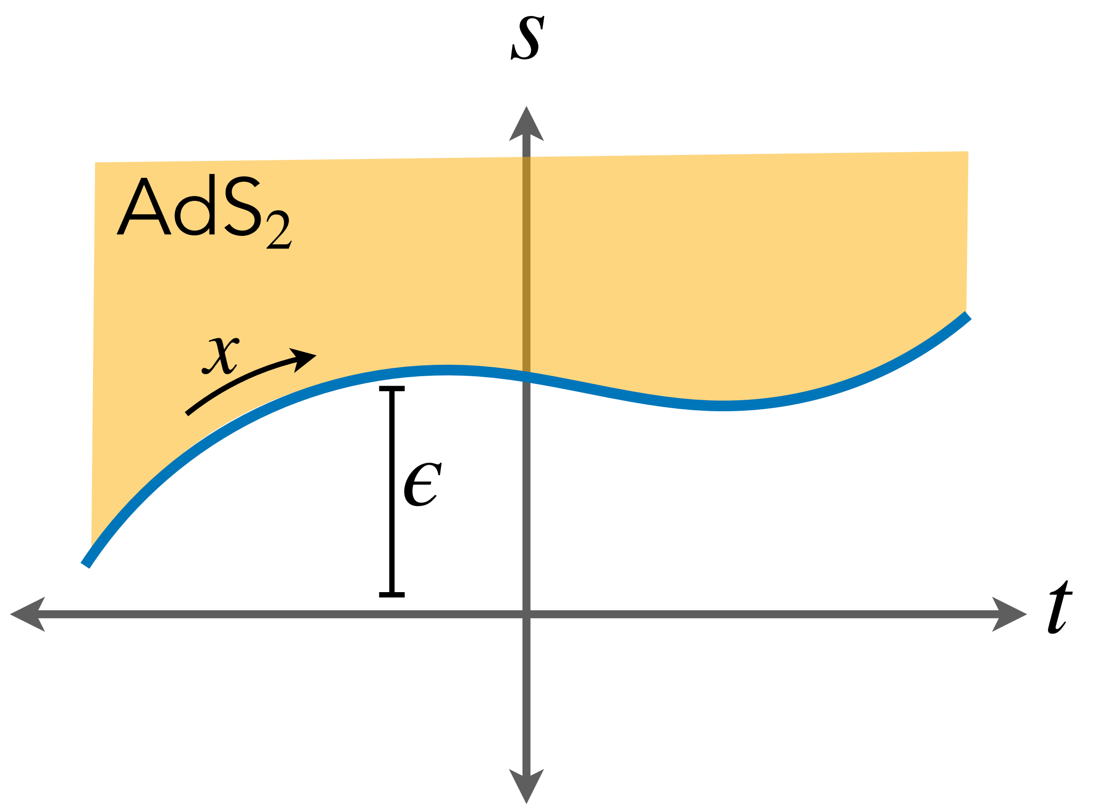
What about a non- AdS_2 string in AdS_3 or in $AdS_3 \times S^1$?



Thank you for listening

Extra slides

Lightning review of JT Gravity



Action:

$$S_{\rm JT}[\phi, h] = S_{\rm Gauss-Bonnet} - \frac{1}{16\pi G_N}\left[\int_{\mathcal{M}} d^2\sigma\sqrt{h}\phi(R+2) + 2\int_B dx\sqrt{h_{xx}}\phi(K-1)\right],$$

$$S_{\rm m}[y, h] = \int_{\mathcal{M}} d^2\sigma\frac{\sqrt{h}}{2}\left(h^{\alpha\beta}\partial_\alpha y\partial_\beta y + m^2y^2\right),$$

Boundary conditions along a boundary curve $(s(x), t(x))$:

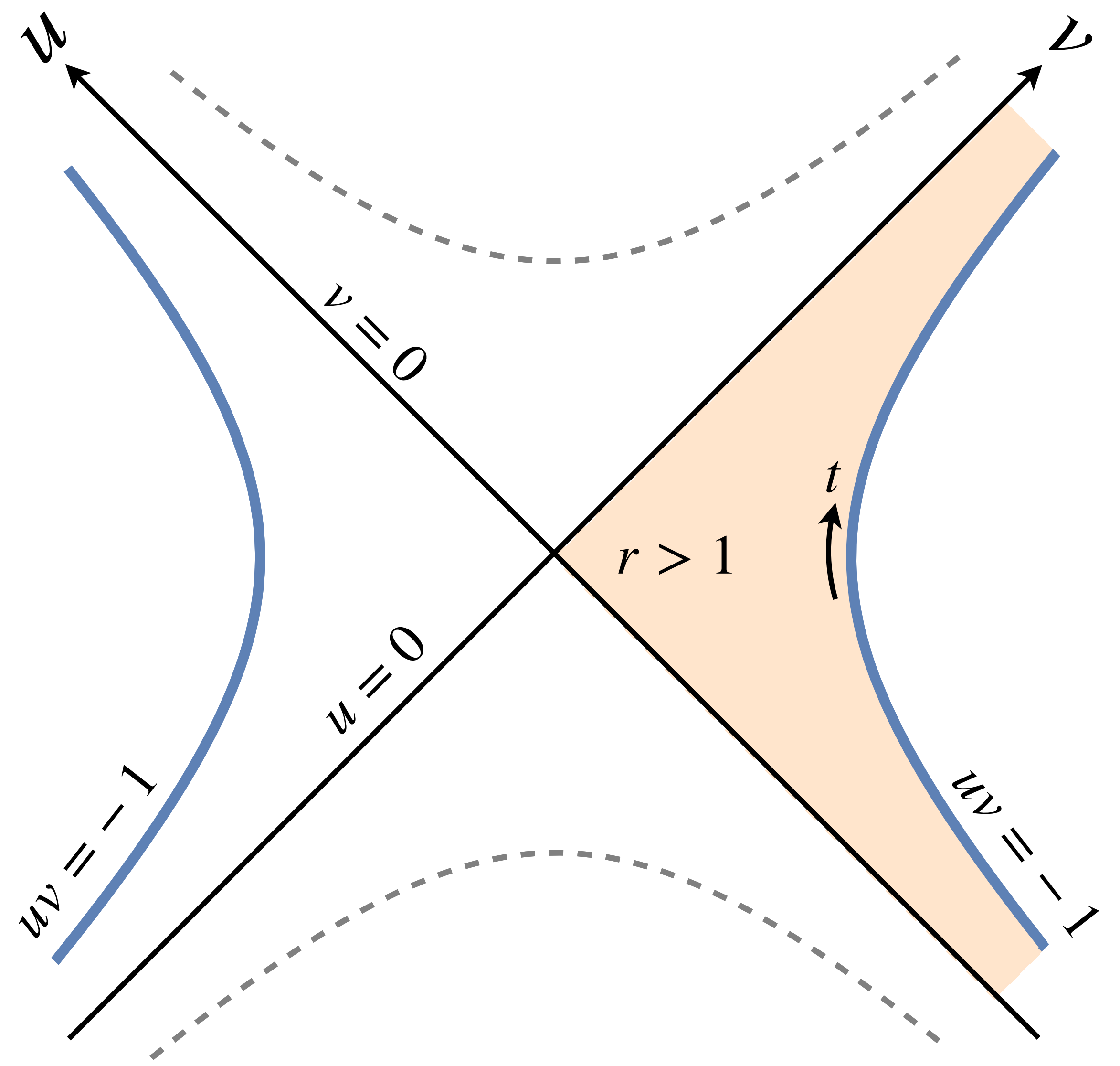
$$h_{xx} = \frac{\dot{s}(x)^2 + \dot{t}(x)^2}{s^2(x)} = \frac{1}{\epsilon^2}, \qquad \phi(s(x), t(x)) = \frac{\tilde{\phi}}{\epsilon}, \qquad y(s(x), t(x)) = \frac{\tilde{y}(x)}{\epsilon^{\Delta-1}}$$

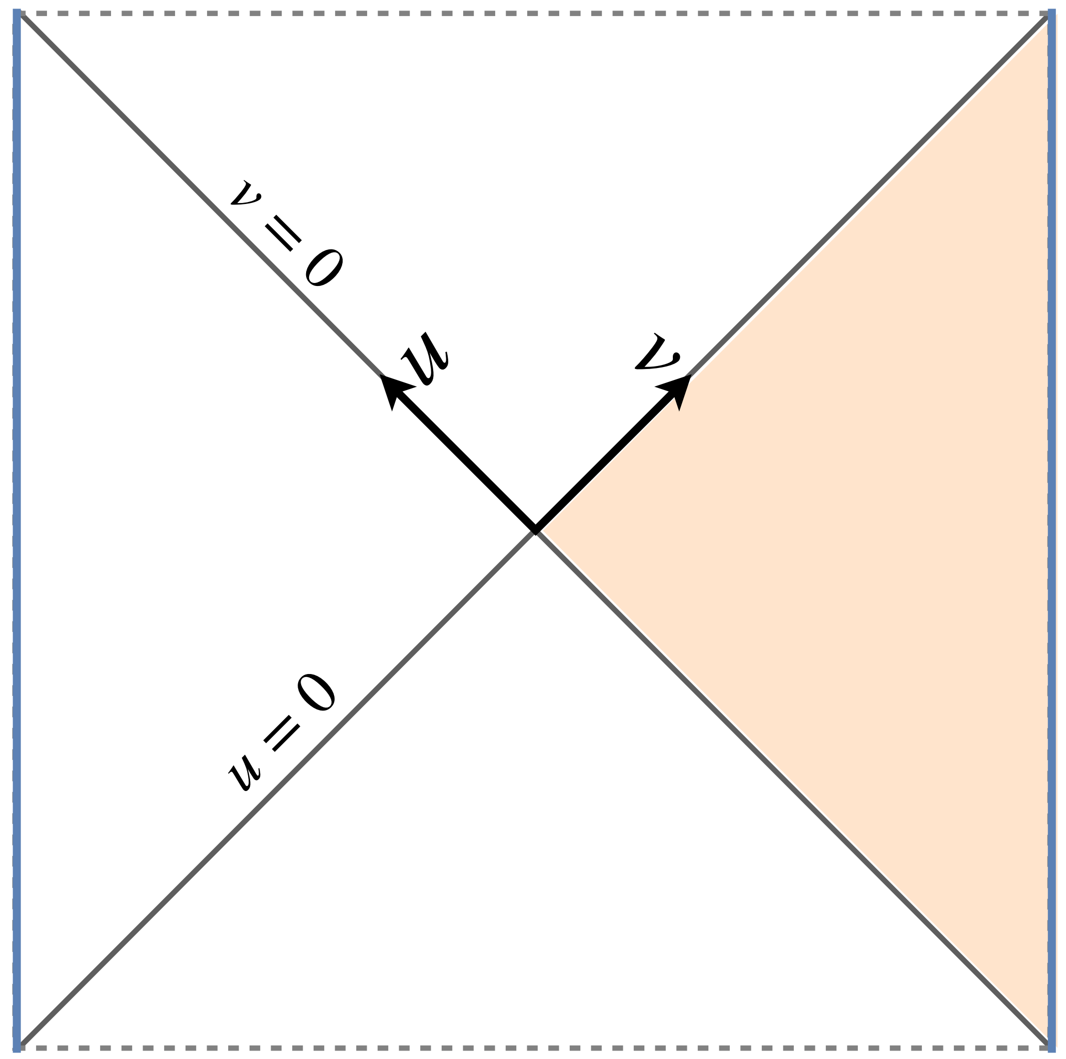
The metric BC implies: $s(x) = \epsilon\dot{t}(x) + O(\epsilon^2)$. Note also that: $R = -2$ and $K = 1 + \epsilon^2\{t, x\} + O(\epsilon^4)$

Wilson loops as versatile observables

- Higher representations dual to D-branes [Drukker, Fill '05; Yamaguchi '06; Gomis, Passerini '06, …]
- Different contours preserving different fractions of superconformal symmetry [Zarembo '02; Drukker, Giombi, Ricci, Trancanelli '07]
- Can study the anomalous dimensions of cusps [Correa, Henn, Maldacena, Sever '12; Correa, Maldacena, Sever '12; Drukker '12; Gromov, Levkovich-Masyluk '15]

Kruskal and Schwarzschild coordinates on AdS$_2$





$$ds^2 = -\frac{4dudv}{(1+uv)^2} = -(r^2-1)dt^2 + \frac{dr^2}{r^2-1}$$

$$u = -\sqrt{\frac{r-1}{r+1}}e^{-t}, \quad v = \sqrt{\frac{r-1}{r+1}}e^{t}$$

Virasoro constraint = extremization over reparametrizations

Let $X^\mu(\sigma)$ solve the EOM with BC $X^\mu(0,t) = \tilde{X}^\mu(\alpha(t))$. Claim: if $\alpha(t) \to \alpha(t) + \delta\alpha(t)$, the variation in the action is:

$$\delta S = \int dt T_{st}(0,t) \frac{\delta\alpha(t)}{\dot{\alpha}(t)} .$$

Let $\bar{X}^\mu(\sigma)$ solve the EOM subject to $\bar{X}^\mu(0,t) = \tilde{X}^\mu(\alpha(t) + \delta\alpha(t))$. It can be represented as $\bar{X}^\mu(\sigma) = X^\mu(\sigma + \delta\sigma)$.

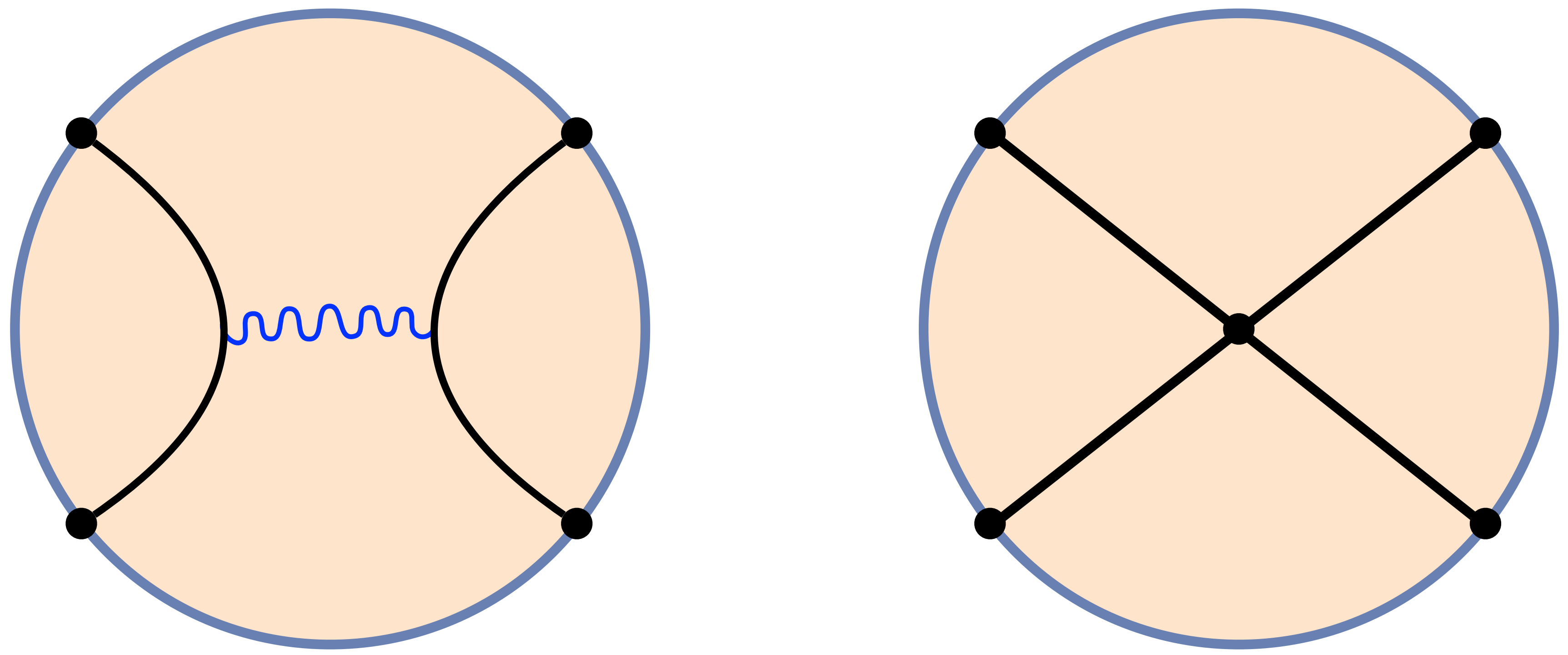
It follows that $\delta s(0,t) = 0$ and $\delta t(0,t) = -\dfrac{\delta\alpha(t)}{\dot{\alpha}(t)}$. The variation in the action is:

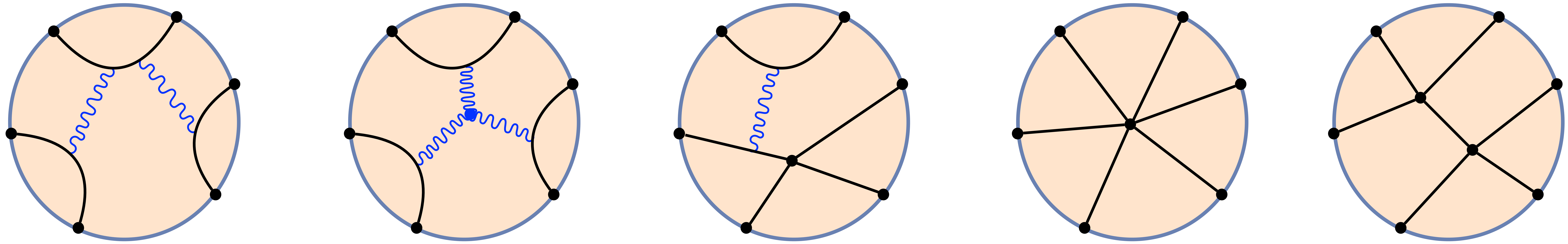
$$\delta S = \int d^2\sigma \frac{\delta S}{\delta X^\mu} \delta X^\mu = -\frac{1}{2} \int d^2\sigma T_{\alpha\beta} \delta h^{\alpha\beta}$$

Where $\delta X^\mu = \delta\sigma^\alpha \partial_\alpha X^\mu(\sigma)$ and $\delta h^{\alpha\beta} = -\partial^\alpha \delta\sigma^\beta - \partial^\beta \delta\sigma^\alpha$. This leads to:

$$\delta S = \int d^2\sigma T_{\alpha\beta} \partial^\alpha \delta\sigma^\beta = \int d^2\sigma \partial^\alpha T_{\alpha\beta} \delta\sigma^\beta - \int dt T_{s\beta} \delta\sigma^\beta$$

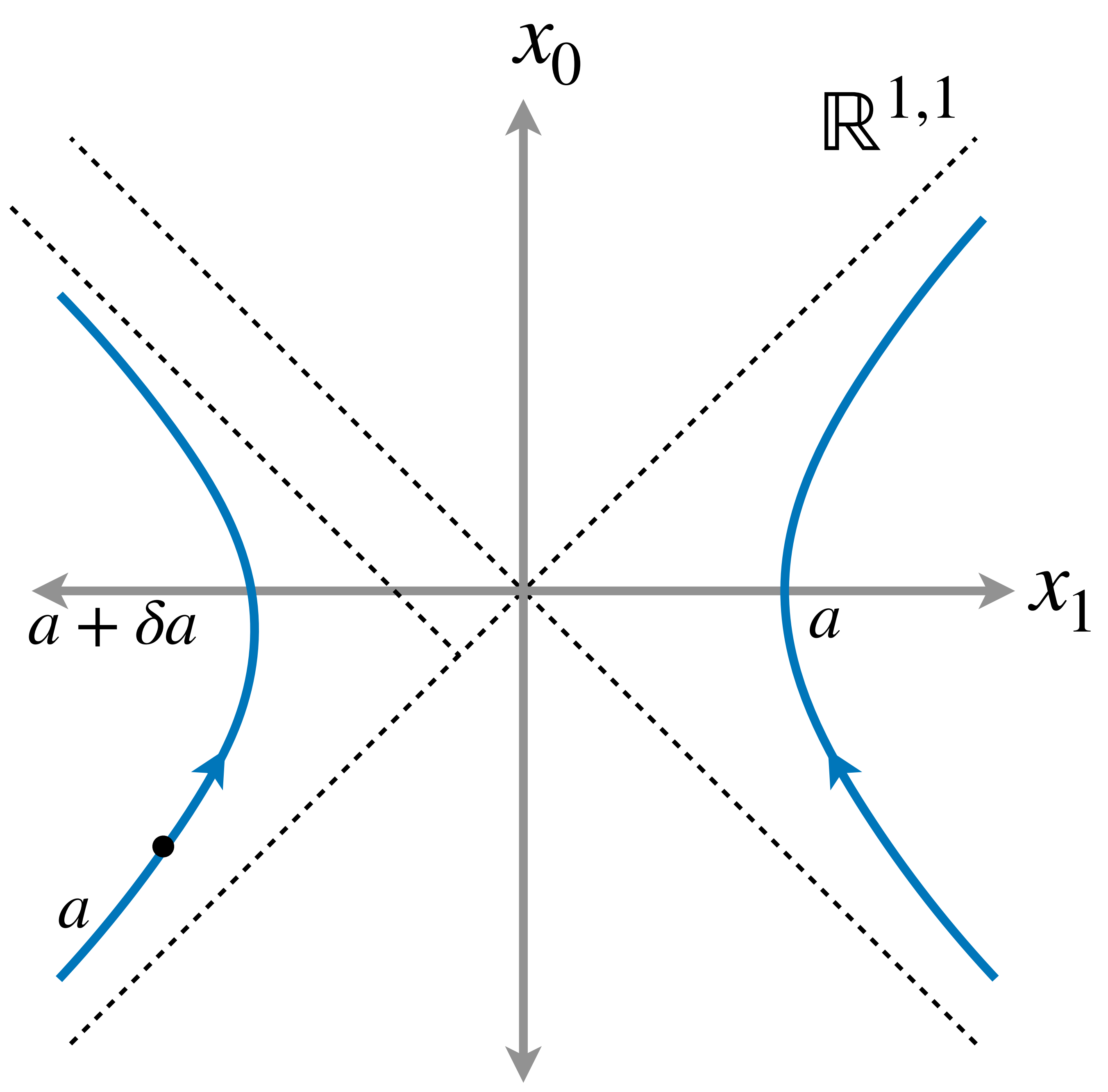
Tree-level 4 and 6-point functions with multiple scalars





OTOC on the string via "shockwaves"

[Murata '17]



OTOC on the string via "shockwaves"

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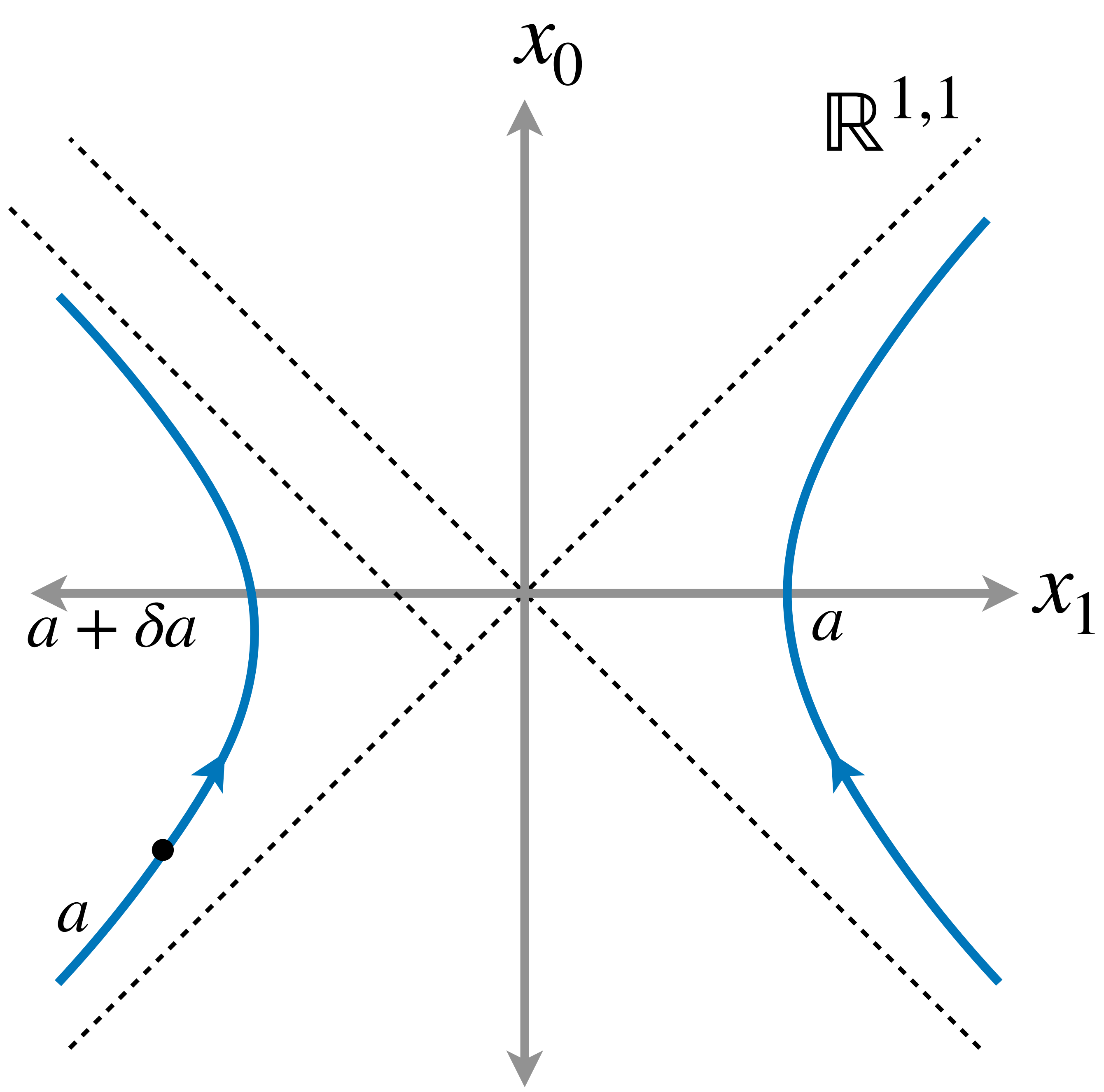
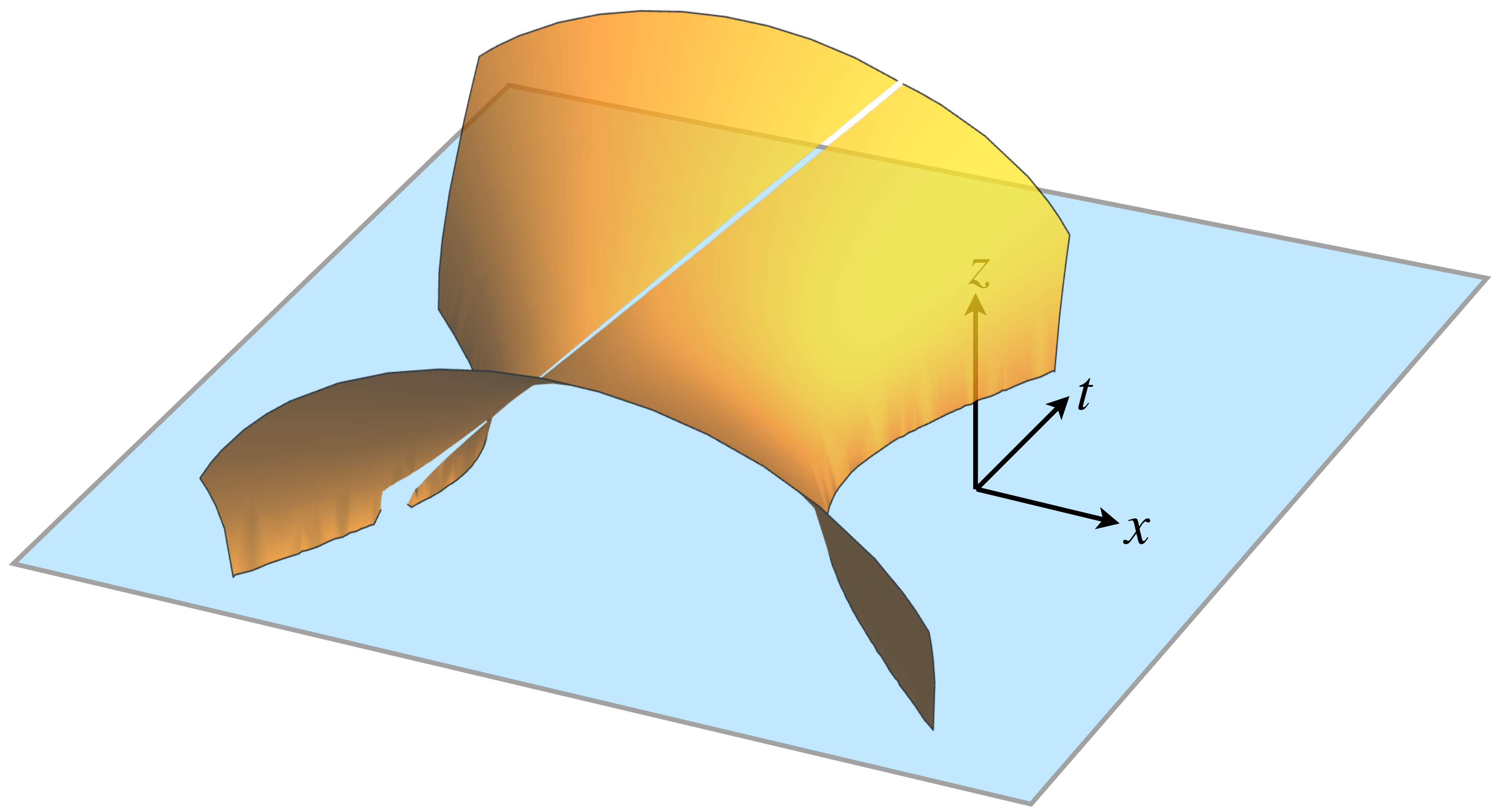
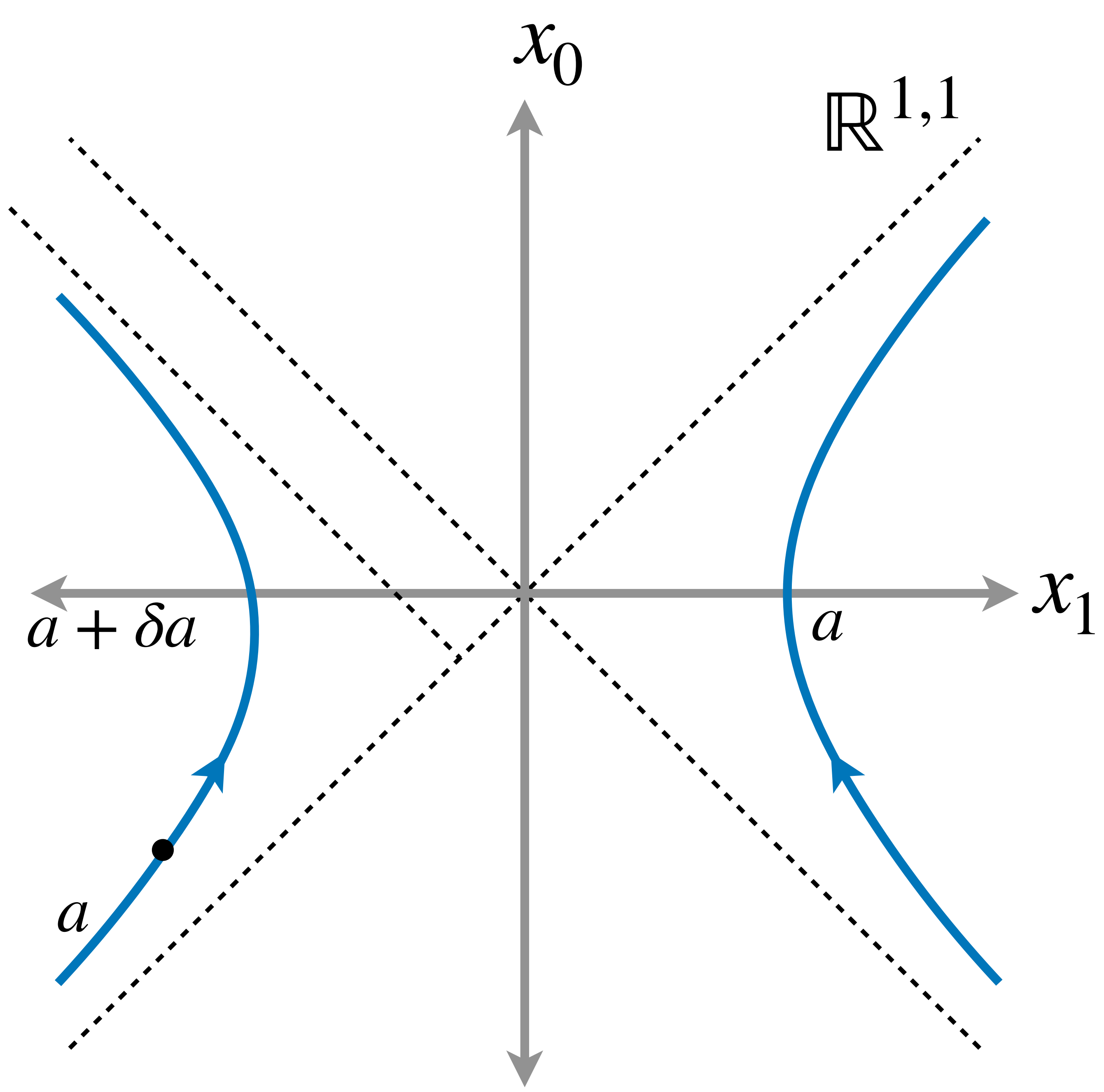


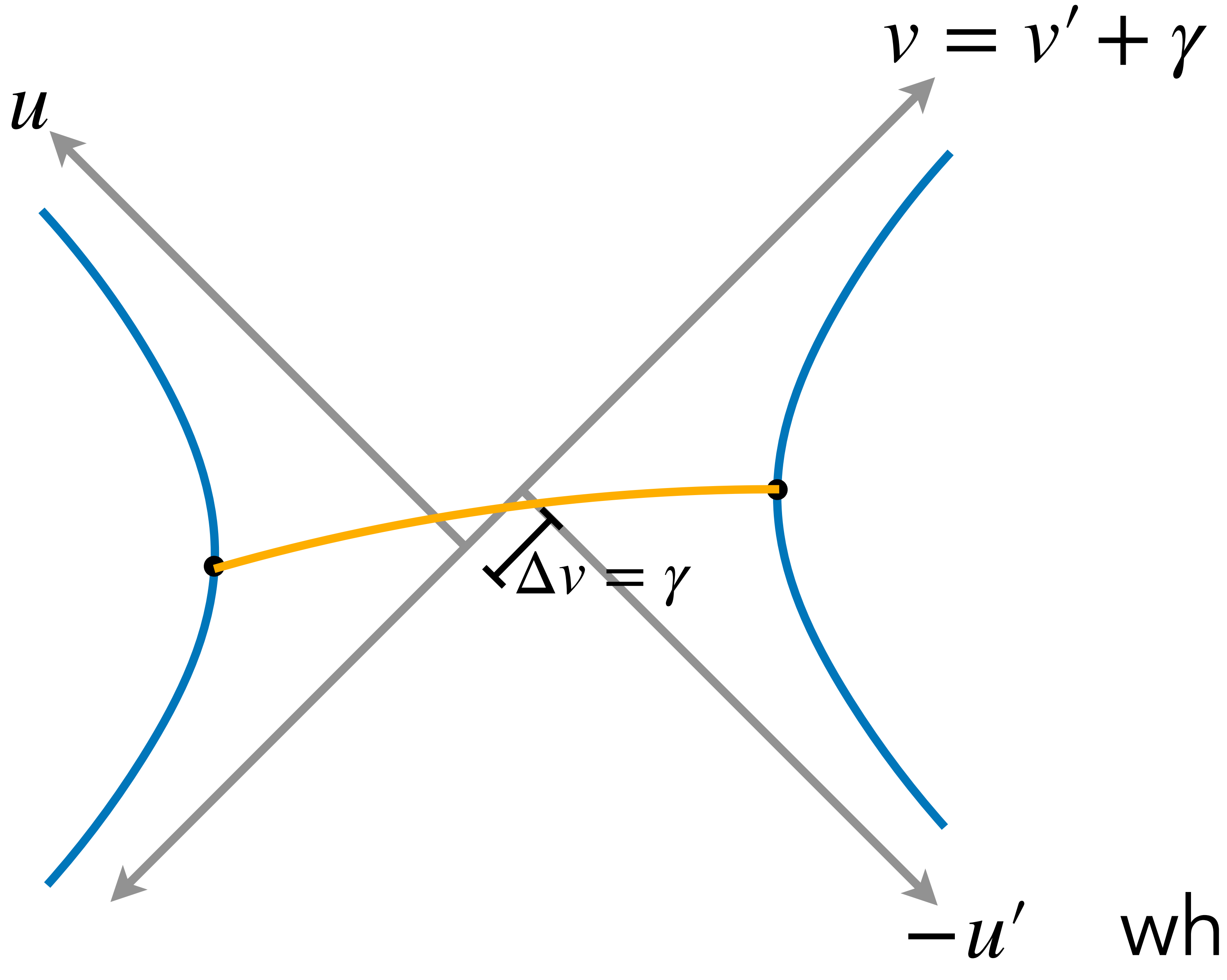
[Vegh '15 - '19]



OTOC on the string via "shockwaves"

[Murata '17]

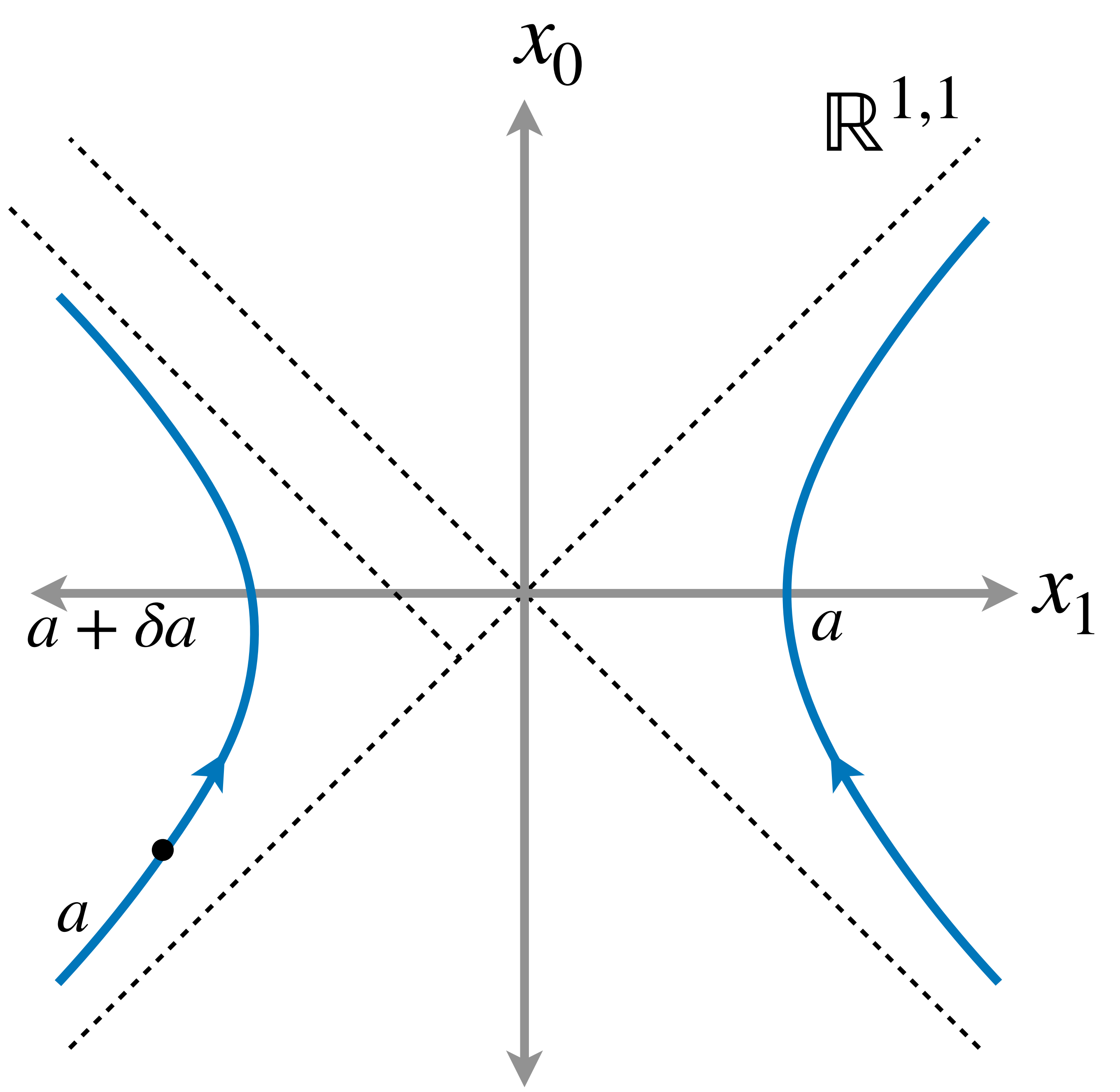


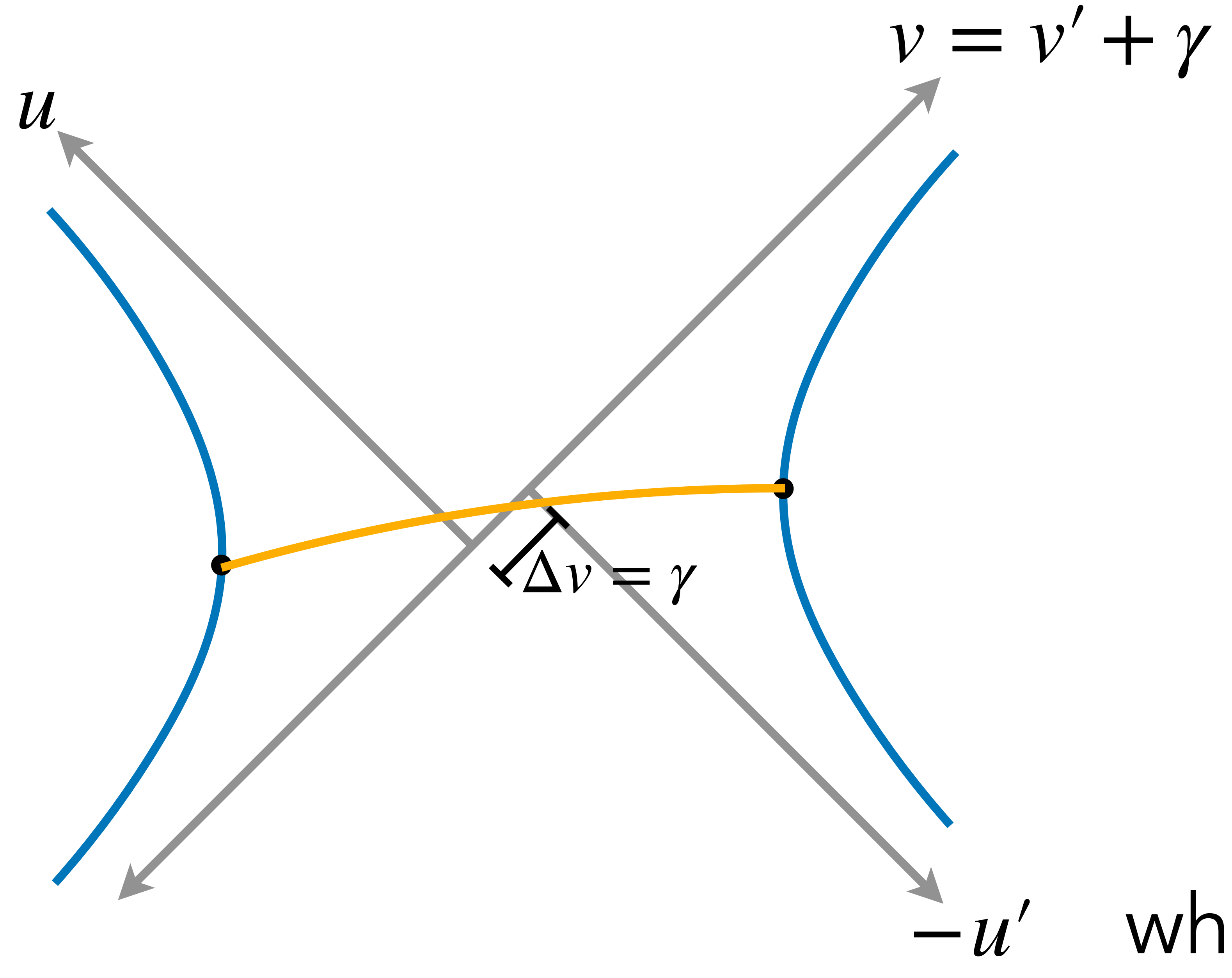


where $\gamma = \dfrac{\delta a}{2a} e^{at}$

OTOC on the string via "shockwaves"

[Murata '17]



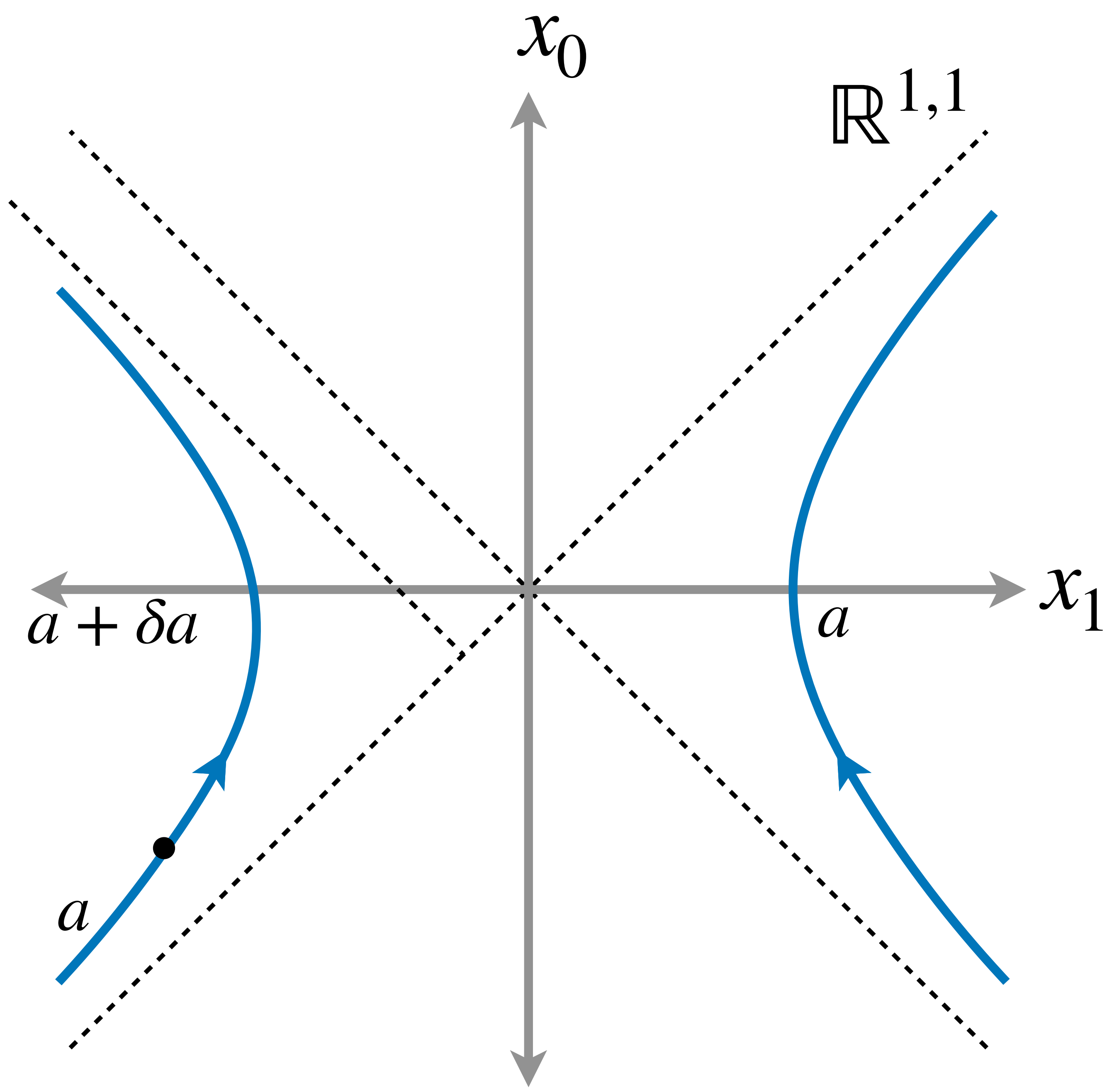


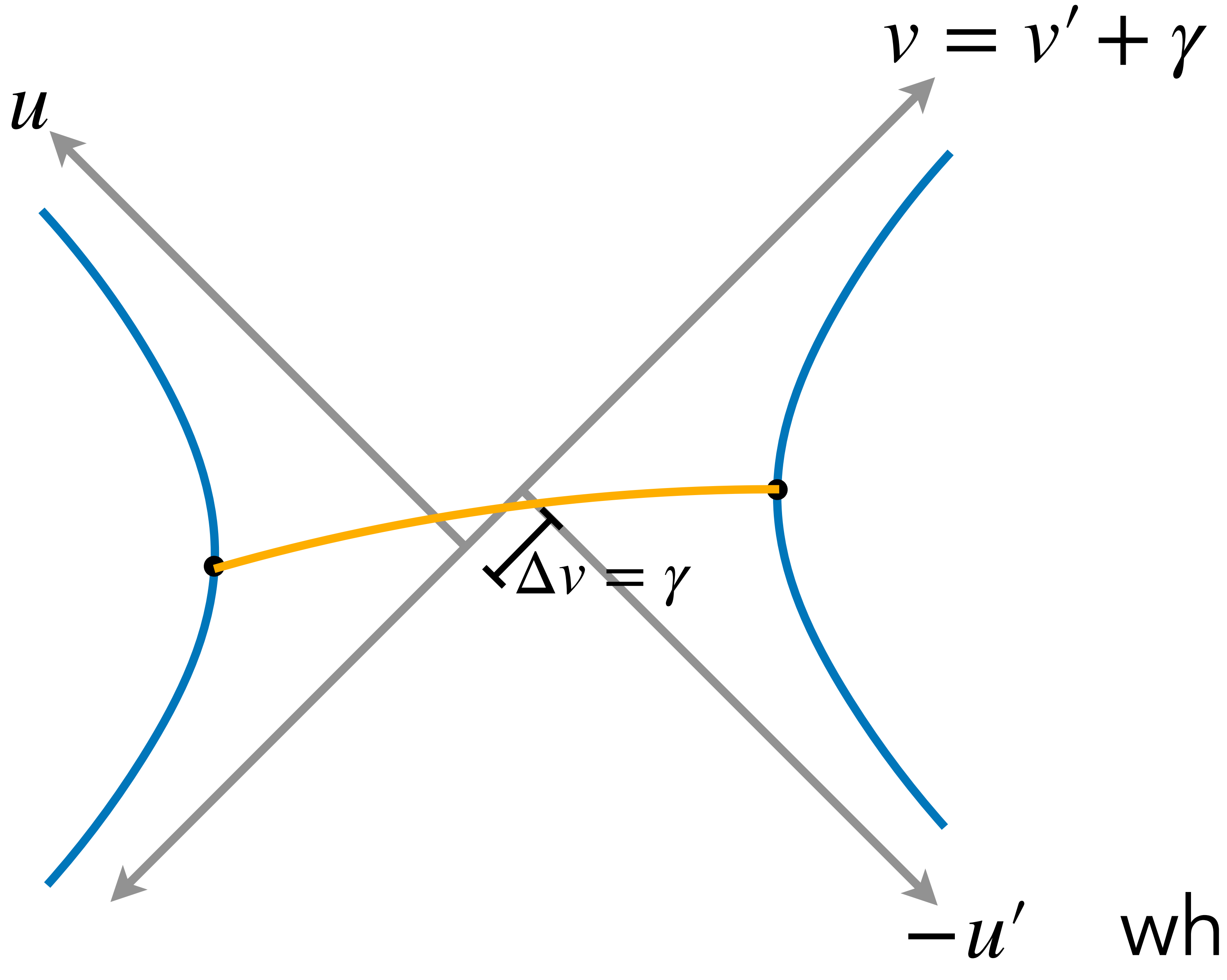
where $\gamma = \dfrac{\delta a}{2a} e^{at}$

$$\frac{\langle V W(t) V W(t)\rangle}{\langle VV\rangle\langle WW\rangle} = \frac{\langle W(t) V_R V W(t)\rangle}{\langle V_R V\rangle\langle WW\rangle} \sim e^{-\Delta_V \ell}$$

OTOC on the string via "shockwaves"

[Murata '17]





where $\gamma = \dfrac{\delta a}{2a} e^{at}$

$$\frac{\langle VW(t)VW(t)\rangle}{\langle VV\rangle\langle WW\rangle} = \frac{\langle W(t)V_R VW(t)\rangle}{\langle V_R V\rangle\langle WW\rangle} \sim e^{-\Delta_V \ell} = \frac{1}{(1+\frac{\gamma}{2})^{2\Delta_V}} = 1 - \frac{\Delta_V \delta a}{2a} e^{at} + \dots$$

and $\beta = \dfrac{2\pi}{a}$